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DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

B.Sc., Mathematics (I Year)

ANALYTICAL GEOMETRY

(Two & Three Dimensions)

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B.Sc., Mathematics – I YEAR

JMMA21 - ANALYTICAL GEOMETRY (Two & Three Dimensions)

SYLLABUS

Unit-I:

Pole, Polar - conjugate points and conjugate lines – diameters – conjugate diameters of an ellipse - semi diameters- conjugate diameters of hyperbola.

Unit-II:

Polar coordinates: General polar equation of straight line – Polar equation of a circle given a diameter, Equation of a straight line, circle, conic – Equation of chord, tangent, normal.

Unit-III:

System of Planes-Length of the perpendicular–Orthogonal projection.

Unit-IV:

Representation of line–angle between a line and a plane – co – planar lines–shortest distance between two skew lines –length of the perpendicular.

Unit-V:

Equation of a sphere-general equation-section of a sphere by a plane-equation of the circle- tangent plane- angle of intersection of two spheres- condition for the orthogonality.

Recommended Text:

1. T.K. Manicavachagam Pillay & T. Natarajan, Analytical geometry (Part-I – Two dimensions), S. Viswanathan (Printers and Publishers) Pvt. Ltd. (2012).
2. T.K. Manicavachagam Pillay & T. Natarajan, Analytical geometry (Part-II – Three dimensions), S. Viswanathan (Printers and Publishers) Pvt. Ltd. (2012).

3. S. Arumugam and A. Thangapandi Issac, Analytical geometry 3D and Vector Calculus, New Gamma Publishing House, Palayamkottai, 2011.

JEMA11 - ANALYTICAL GEOMETRY (Two & Three Dimensions)

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Unit-I:

Pole, Polar - conjugate points and conjugate lines – diameters – conjugate diameters of an ellipse - semi diameters- conjugate diameters of hyperbola.

1.1 polar and pole:

Definition:

A polar of a point with regard to a conic is the locus of a point of intersection of tangents at the extremities of chords to that point. The point is called the pole of locus.

Let P be any point inside (or) outside the conic. Draw any chord AB and $A'B'$ passing through P if a tangents to the conic at A and B meet at Q and the tangents to the conic at A' and B' meet at Q'. Then the line QQ' is called polar with P as its pole.

1.2 Conjugate points and conjugate lines:

It the polar of P with respect to a conic passes through Q, then the polar of Q will passes through P and such points are said to be conjugate points.

If the pole of the line $l_1$with respect to a conic lies on the another line l_2, then the pole of second line will lie on the first line and such lines are called conjugate lines.

Definition:

If two points are such that the polar of one passes through the other, the points are called conjugate points.

If two lines are such that the pole of each lies on the anther, the two lines are called conjugate lines.

1.2.1 The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) on it.

Let P be the given point (x_1, y_1) and $Q(x_2, y_2)$ be a point on the ellipse very close to P.

The equation to PQ is

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1) \qquad \dots\dots\dots(1)$$

Since P and Q lie on the ellipse.

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \;\;\&\;\; \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1$$

By subtraction,

$$\frac{x_1^2-x_2^2}{a^2}+\frac{y_1^2-y_2^2}{b^2}=0$$

$$i.e., \qquad \frac{(x_1+x_2)(x_1-x_2)}{a^2}=-\frac{(y_1+y_2)(y_1-y_2)}{b^2}$$

$$\therefore \frac{y_1-y_2}{x_1-x_2}=\frac{-b^2}{a^2}\cdot\frac{x_1+x_2}{y_1+y_2}$$

Substituting this value in the equation of PQ, we get,

$$y-y_1=\frac{-b^2}{a^2}\cdot\frac{x_1+x_2}{y_1+y_2}(x-x_1) \quad \dots\dots\dots(2)$$

When Q tends to P, in the limiting position the secant becomes the tangent at P and y_2 becomes equal to y_1.

Hence putting $y_2=y_1$the equation of the tangent at P becomes.

$$y-y_1=\frac{-b^2}{a^2}\cdot\frac{x_1}{y_1}(x-x_1)$$

$$i.e., \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

Hence the equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$

1.2.2. The chord of contact of tangent drawn from. (x_1, y_1) to the ellipse $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$

Let the points of contact of tangent drawn from the point (x_1, y_1) to the ellipse be P(x_2, y_2) and Q(x_3, y_3).

The tangent at P is $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1.$

It passes through (x_1, y_1)

$$\therefore \quad \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} = 1 \quad \ldots \ldots (1)$$

The tangent at Q is

$$\frac{xx_3}{a^2} + \frac{yy_3}{b^2} = 1$$

It passes through (x_1, y_1)

$$\therefore \quad \frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} = 1 \quad \ldots \ldots (2)$$

The equation (1) and (2) show that the co-ordinates (x_2, y_2) and (x_3, y_3) satisfy the equation $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$

Hence P and Q lies on the line $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$

$\therefore$ The equation of PQ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$

1.2.3. The polar of the point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Let PQ be any chord passing through the point (x_1, y_1) and let the tangents at P, Q interest at (h, k).

The locus of (h, k) is the polar of (x_1, y_1) , PQ is the chord of contact of tangent drawn from the point (h, k).

∴ Its equation is

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1$$

It passes through (x_1, y_1)

$$\frac{x_1 h}{a^2} + \frac{y_1 k}{b^2} = 1$$

∴ locus of (h, k) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1,$ which is the polar of the point (x_1, y_1).

Corollary:

The polar of the focus is the corresponding directrix co-ordinates of S are (ae, o).

∴ polar of (ae, o) is $\frac{x.ae}{a^2} = 1$

$$i.e., x = \frac{a}{e}.$$

This is the equation of the directrix corresponding to the focus S.

1.2.4. The pole of the line $lx + my + n = 0$ with respect to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Let (x_1, y_1) be the pole of the line $lx + my + n = 0$.

The polar of (x_1, y_1) with respect to the ellipse is

$$\frac{x - x_1}{a^2} + \frac{y - y_1}{b^2} - 1 = 0$$

This must be identical with $lx + my + n = 0$

$$\therefore \frac{x_1}{a^2 l} = \frac{y_1}{b^2 m} = -\frac{1}{n}$$

$\therefore$ The pole is $\left(-\frac{la^2}{n}, -\frac{mb^2}{n}\right)$.

1.2.5. The condition for the lines $lx + my + n = 0, l_1 x + m_1 y + n_1 = 0$ to be conjugate with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The pole of the line $lx + my + n = 0$ with respect to the ellipse is $\left(-\frac{la^2}{n}, -\frac{mb^2}{n}\right)$

If it lies on the line $l_1 x + m_1 y + n_1 = 0$, the two lines are conjugate.

$$\therefore -\frac{ll_1 a^2}{n} - \frac{mm_1 b^2}{n} + n_1 = 0.$$

$(-)$x$\Rightarrow$ $\quad ll_1 a^2 + mm_1 b^2 = nn_1$.

Example 1:

Chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touch $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$. Find the locus of their poles.

Solution:

Let (x_1, y_1) be the pole of PQ, a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$.

PQ is the polar of (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The equation of PQ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$

$$\therefore y = -\frac{b^2 x_1}{a^2 y_1} . x + \frac{b^2}{y_1}$$

It touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$

$$\therefore \frac{b^4}{y_1^2} = \alpha^2 . \frac{b^4 x_1^2}{a^4 y_1^2} + \beta^2$$

$$(i.e.,)\ x_1^2 \alpha^2 b^4 + y_1^2 \beta^2 a^4 = a^4 b^4$$

$$\div\ a^4 b^4 \Rightarrow \frac{x_1^2 \alpha^2}{a^4} + \frac{y_1^2 \beta^2}{b^4} = 1.$$

Another method:

Let (x_1, y_1) be middle point,

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$y = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) \frac{b^2}{y_1} - \frac{b^2 x_1}{a^2 y_1} . x.$$

$$y = mx + c$$

Condition for line $y = mx + c$ touches ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ is $c^2 = \alpha^2 m^2 + \beta^2$

$$\left[\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right)\left(\frac{b^2}{y_1}\right)\right]^2 = \alpha^2 \left(\frac{-b^2 x_1}{a^2 y_1}\right)^2 + \beta^2$$

$$\Rightarrow \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right)^2 = \alpha^2 \frac{x_1^4}{a^4} + \beta^2 \frac{y_1^2}{b^4}.$$

Locus of (x_1, y_1) is

$$\alpha^2 \frac{x^2}{a^4} + \beta^2 \frac{y^2}{b^4} = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$$

$$\frac{x^2\alpha^2}{a^4} + \frac{y^2\beta^2}{b^4} = 1 \quad Since\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\therefore$ Locus of (x_1, y_1) is

$$\frac{x^2\alpha^2}{a^4} + \frac{y^2\beta^2}{b^4} = 1$$

Example 2:

A chord PQ of an ellipse subtends a right angle at the centre of the ellipse. Show that the locus of intersection of the tangents at P and Q is the ellipse $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$.

Solution:

Let the tangent at P and Q intersect at (x_1, y_1), PQ is the chord of contact of tangent drawn from (x_1, y_1).

$$\therefore PQ\ is\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

C is the centre of the ellipse and also the origin of co-ordinates.

The combined equation of CP and CQ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\right)^2$$

$$i.e.,\ x^2\left(\frac{1}{a^2}-\frac{x_1^2}{a^4}\right)-\frac{2x_1y_1xy}{a^2b^2}+y^2\left(\frac{1}{b^2}-\frac{y_1^2}{b^4}\right)=0.$$

PQ subtends an angle 90° at C.

∴ Coefficient of x^2 + coefficient of $y^2=0$

$$\frac{1}{a^2}-\frac{x_1^2}{a^4}+\frac{1}{b^2}-\frac{y_1^2}{b^4}=0$$

$$\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Locus of (x_1,y_1) is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}$$

Example 3:

Show that the conjugate lines through a focus of an ellipse are at right angles.

Solution:

Let the conjugate lines be $lx+my+n=0,\ \&\ l_1x+m_1y+n_1=0\ \ldots\ldots..(1)$

Since they are conjugate lines to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$$\therefore\ a^2ll_1+b^2mm_1=nn_1 \qquad \ldots\ldots(2)$$

Given conjugate lines passes through focus $(ae,0)$

Substitute, $x=ae, y=0\ \ in\ (1)$

$$(1)\Rightarrow\ \ lae+0+n=0\ \ ;\ \ l_1ae+n_1=0$$

$$\Rightarrow\ \ lae=-n\ \ ;\ \ l_1ae=-n_1$$

$$\Rightarrow \quad (lae)(l_1 ae) = nn_1$$

Substitute in equation (2)

$$\Rightarrow \quad a^2 ll_1 + b^2 mm_1 = ll_1 a^2 e^2$$

$$\Rightarrow \quad a^2 ll_1 - ll_1 a^2 e^2 + b^2 mm_1 = 0$$

$$\Rightarrow \quad a^2 ll_1 (1 - e^2) + b^2 mm_1 = 0$$

$$\Rightarrow \quad a^2 ll_1 + b^2 mm_1 = 0$$

$$\Rightarrow \quad b^2 (ll_1 + mm_1) = 0$$

$$ll_1 + mm_1 = 0$$

$$ll_1 = -mm_1$$

$$\frac{ll_1}{mm_1} = -1$$

$$lx + my + n = 0, \& \; l_1 x + m_1 y + n_1 = 0 \; is - 1$$

∴ The two lines are at right angles.

1.2.6. The equation of the pair of tangents drawn from the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equation of any line passing through the point (x_1, y_1) is of the form

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \quad \dots\dots (1)$$

$$\therefore x = x_1 + r\cos\theta$$

$$y = y_1 + r\sin\theta$$

If this point lies on the ellipse,

$$\frac{(x_1 + r\cos\theta)^2}{a^2} + \frac{(y_1 + r\sin\ \theta)^2}{b^2} = 1$$

$$(i.e.,) r^2\left(\frac{cos^2\ \theta}{a^2} + \frac{sin^2\ \theta}{b^2}\right) + 2r\left(\frac{x_1\cos\theta}{a^2} + \frac{y_1\sin\theta}{b^2}\right) + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0 \quad \ldots (2)$$

This equation in r gives two values $r_1\ and\ r_2$. If the line (1) meets the ellipse at P and Q, then the length OP and OQ are the roots r_1, r_2 of the equation (2)

If the line is a tangent to the ellipse, P and Q should coincide,

$$(i.e.,) r_1 = r_2$$

Equation (2) should have equal roots.

$$\left(\frac{x_1\cos\theta}{a^2} + \frac{y_1\sin\theta}{b^2}\right)^2 = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)\left(\frac{cos^2\ \theta}{a^2} + \frac{sin^2\ \theta}{b^2}\right) \quad \ldots\ldots (3)$$

From this equation we can find the directions of the tangents drawn to the ellipse from (x_1, y_1).

Eliminate θ between the equations (1) & (3) and the resulting equation will be the equation of the pair of tangents from (x_1, y_1).

$$\left\{\frac{x_1(x - x_1)}{a^2} + \frac{y_1(y - y_1)}{b^2}\right\}^2 = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)\left[\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{b^2}\right]$$

$$(i.e.,)\left\{\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right) - \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right)\right\}^2$$

$$= \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)\left\{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 2\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right) + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right\}$$

If we assume that

$$T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

and $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1,$

then, the equation becomes,

$$(T - S_1)^2 = S_1(S - 2T + S_1)$$

$$(i.e.,)\ T^2 = SS_1.$$

Corollary:

The equation of the directrix circle can also be found from this equation.

The combined equation of the pair of tangents drawn from the point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$

If the angle between the tangents is 90°,

Coefficient of x^2 + coefficient of $y^2 = 0$

$$\therefore\ \frac{1}{a^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) - \frac{x_1^2}{a^4} + \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) - \frac{y_1^2}{b^4} = 0$$

$$i.e.,)\ \frac{x_1^2 + y_1^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

$$i.e.,)\ x_1^2 + y_1^2 = a^2 + b^2$$

$\therefore$ Locus of the point (x_1, y_1) is the circle

$$x^2 + y^2 = a^2 + b^2$$

Example 4:

A pair of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercept on the x-axis a constant length c. Prove that the locus of the point of intersection is the curve

$$4y^2(b^2x^2+a^2y^2-a^2b^2)=\ c^2(y^2-b^2)^2.$$

Solution:

Let the tangents intersect at (x_1,y_1)

The combined equation of the tangents drawn from $(x_1,y_1)$is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2 \ldots .(1)$$

The x-co-ordinates of the points,where these tangents meet the x-axis are got by putting $y=0$ in the equation.

From equation (1) sub in $y=0$

$$\left(\frac{x^2}{a^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-1\right)^2$$

$$\frac{x^2}{a^2}\left(\frac{x^2}{a^2}+\frac{y_1^2}{b^2}-1\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\frac{x^2x_1^2}{a^4}-\ \frac{2xx_1^2}{a^2}+1$$

$$\frac{x^2}{a^2}\left(\frac{y_1^2}{b^2}-1\right)+\frac{2xx_1}{a^2}-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1+1\right)=0.$$

$$\frac{x^2}{a^2}\left(\frac{y_1^2}{b^2}-1\right)+\frac{2xx_1}{a^2}-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)=0.$$

Multiply by a^2.

$$\Rightarrow\ x^2\left(\frac{y_1^2}{b^2}-1\right)+2xx_1-a^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)=0.$$

This equation is quadratic form $ax^2+bx+c=0$

Sum of the roots

$$\alpha+\beta=\frac{-b}{a}=-\frac{2x_1}{\left(\frac{y_1^2}{b^2}-1\right)}$$

Product of the roots

$$\alpha\beta=\frac{c}{a}=\frac{-a^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)}{\left(\frac{y_1^2}{b^2}-1\right)}$$

The intercept on the x-axis is constant c.

$$\therefore c=\alpha-\beta$$

$$c^2=(\alpha-\beta)^2$$

$$=(\alpha-\beta)^2-4\alpha\beta$$

$$=\frac{4x_1^2}{\left(\frac{y_1^2}{b^2}-1\right)^2}+\frac{4a^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)}{\frac{y_1^2}{b^2}-1}$$

$$c^2=\frac{4x_1^2b^4}{(y_1^2-b^2)^2}+4a^2b^2\frac{\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)}{(y_1^2-b^2)}$$

Multiply by $(y_1^2-b^2)^2 \Rightarrow$

i.e.,) $c^2(y_1^2-b^2)^2=4x_1^2b^4+4a^2b^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)(y_1^2-b^2)$

Since $4a^2b^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)(y_1^2-b^2)$

$$=4x_1^2y_1^2b^2-4x_1^2b^4+4y_1^2a^2-4y_1^2a^2b^2$$

$$c^2(y_1^2-b^2)^2=4y_1^2(b^2x_1^2+a^2y_1^2-a^2b^2)$$

$\therefore$ Locus of (x_1,y_1) is the curve.

$$c^2(y^2-b^2)^2=4y^2(b^2x^2+a^2y^2-a^2b^2).$$

1.2.7. The locus of the middle points of a series of parallel chords of an ellipse.

Let (x_1, y_1) be the mid point of one of the parallel chords. Then the equation of the chords is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

Since the chords are parallel, their slopes are constant, say m.

$$\therefore -\frac{b^2x_1}{a^2y_1} = m.$$

$\therefore$ Locus of (x_1, y_1) is $y = -\frac{b^2}{a^2m}x,$

which is a straight line through the centre of the ellipse.

Thus, the locus of the middle points of the chords of the ellipse parallel to $y = mx$ is the line $y = -\frac{b^2}{a^2m}x.$

If $y = m_1x$ bisects all chords parallel to $y = mx$

$$m_1 = -\frac{b^2}{a^2m}$$

$$(i.e.,)\ mm_1 = -\frac{b^2}{a^2}.$$

From the symmetry of this relation it is apparent that the diameter $y = mx$ will bisect chords parallel to $y = m_1x.$

1.3 Diameters:

1.4 Conjugate diameters of an ellipse:

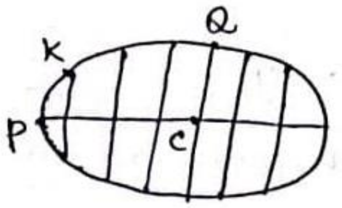
Definition:

Two diameters are said to be conjugate to each other when one bisects chords parallel to the other. Thus, the line $y = mx, y = m_1x$ are conjugate diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $mm_1 = -\frac{b^2}{a^2}$.

1.4.1. The tangents at the extremities of a diameter are parallel to the chords bisected by the diameter.

Let the chords be parallel to the diameter $y = mx$ (CQ).

Then the diameter bisecting those chords is $y = m_1x$ (CP) if $mm_1 = -\frac{b^2}{a^2}$.



Let $P(x_1, y_1)$ be the extremity of this diameter.

Then $y_1 = m_1x_1$

The tangent at P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

$\therefore$ The slope of the tangent $= -\frac{b^2x_1}{a^2y_1}$

$$= -\frac{b^2x_1}{a^2m_1x_1} = -\frac{b^2}{a^2m_1} = m.$$

$\therefore$ The tangent at P is parallel to the parallel chords.

1.4.2. The tangents at the extremities of a chord will intersect on the diameter bisecting the chord.

Let the equation of the chord be $y = mx + c \quad \ldots.(1)$ and the tangents at its extremities intersect at (x_1, y_1)

$\therefore$ The chord of contact of tangent drawn from (x_1, y_1) to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \ldots..(2)$$

Equation (1) & (2) represent the same straight line.

$$\frac{l}{\frac{y_1}{b^2}} = -\frac{m}{\frac{x_1}{a^2}}$$

(ie) $$y_1 = -\frac{b^2}{a^2 m}x_1.$$

$\therefore (x_1, y_1)$ lies on the line $y = -\dfrac{b^2}{a^2 m}x.$

This is the diameter which bisects all chords parallel to $y = mx + c$.

Example 5:

A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$ whose centre is C meets the circle

$x^2 + y^2 = a^2 + b^2$ at Q and Q'. Prove that CQ and CQ' are conjugate diameter of the ellipse.

Solution:

Any tangent to the ellipse is $y = mx + \sqrt{a^2m^2 + b^2}$

C is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$\therefore$ C is the origin and the tangent meets the circle.

$$x^2 + y^2 = a^2 + b^2 \text{ at } Q \text{ and } Q'$$

$\therefore$ The combined equation of CQ and CQ' is

$$x^2 + y^2 = (a^2 + b^2)\left(\frac{y - mx}{\sqrt{a^2m^2 + b^2}}\right)^2$$

$$x^2 + y^2 = (a^2 + b^2)\left(\frac{y^2}{a^2m^2 + b^2} - \frac{2mxy}{a^2m^2 + b^2} + \frac{m^2x^2}{a^2m^2 + b^2}\right)$$

$$x^2 + y^2 = \frac{y^2(a^2 + b^2)}{a^2m^2 + b^2} - \frac{2mxy(a^2 + b^2)}{a^2m^2 + b^2} + \frac{m^2x^2(a^2 + b^2)}{a^2m^2 + b^2}$$

$$\Rightarrow y^2\left(1 - \frac{(a^2 + b^2)}{a^2m^2 + b^2}\right) + \frac{2(a^2 + b^2)mxy}{a^2m^2 + b^2} + x^2\left(1 - \frac{m^2(a^2 + b^2)}{a^2m^2 + b^2}\right)$$

$$i.e.,)\frac{y^2[a^2(m^2 - 1)]}{a^2m^2 + b^2} + \frac{2(a^2 + b^2)mxy}{a^2m^2 + b^2} + \frac{x^2[b^2(1 - m^2)]}{a^2m^2 + b^2} = 0$$

$$\frac{y^2[a^2(m^2 - 1)] + 2(a^2 + b^2)mxy + x^2[b^2(1 - m^2)]}{a^2m^2 + b^2} = 0$$

$$\Rightarrow \; y^2[a^2(m^2 - 1)] + 2(a^2 + b^2)mxy + x^2[b^2(1 - m^2)] = 0$$

$\div \; a^2(m^2 - 1)$

$$\Rightarrow \; y^2 + \frac{2(a^2 + b^2)mxy}{a^2(m^2 - 1)} + \frac{x^2[b^2(1 - m^2)]}{a^2(m^2 - 1)} = 0.$$

$$\Rightarrow \; y^2 + \frac{2(a^2 + b^2)mxy}{a^2(m^2 - 1)} - \frac{b^2}{a^2}x^2 = 0.$$

If the individual equation of the CQ and CQ' are $y - m_1x = 0$ and $y - m_2x = 0$,then from their combined equation, we get,

$$m_1m_2 = -\frac{b^2}{a^2}.$$

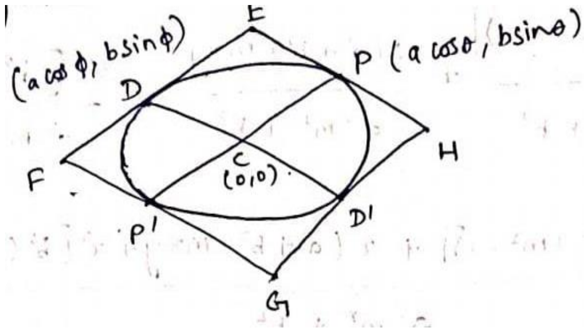
$\therefore y - m_1 x = 0$ and $y - m_2 x = 0$ are conjugate diameters.

$\therefore CQ$ and CQ' are pair of conjugate diameters.

1.5 Semi diameters:

1.5.1 Properties of conjugate diameters:

(1) The eccentric angles of the ends of a pair of conjugate diameters differ by a right angle.



Let θ and ∅ be the eccentric angles of the extremities P and D of two conjugate diameters.

The slopes of CP and CD are $\frac{b \sin\theta}{a\cos\theta}$ and $\frac{b\sin\emptyset}{a\cos\emptyset}$.

Slope of (CP) x slope of CD ⇒

$$\frac{b \sin\theta}{a\cos\theta} \cdot \frac{b\sin\emptyset}{a\cos\emptyset} = -\frac{b^2}{a^2}$$

$$\sin\theta \sin\emptyset = -\cos\theta \cos\emptyset$$

$$\cos\theta \cos\emptyset + \sin\theta\sin\emptyset = 0$$

$$\cos(\theta - \emptyset) = 0$$

$$\theta \sim \emptyset = \text{an odd multiple of } \frac{\pi}{2}.$$

The co-ordinates of the extremities of two semi-conjugate diameters can therefore be written as $(a\cos\theta, b\sin\theta)$, $(-a\,sin\theta, b\cos\theta)$.

(2) The sum of the squares of two conjugate semi- diameters of an ellipse is constant.

$\therefore\ \boldsymbol{CP^2 + CD^2 = a^2 + b^2}$ **=constant**.

(3) If $\boldsymbol{p}$ be the perpendicular on the tangent at $\boldsymbol{P}$ from the centre of an ellipse, then $\boldsymbol{p.CD = ab}$

(4) The tangents at the ends of a pair of conjugate diameters of an ellipse from a parallelogram of constant area.

$$\therefore \|^m\ EFGH = 4ab.$$

(5) The product of the focal distance of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.

$$SP = a - ae\cos\theta$$

$$S'P = a + ae\cos\theta$$

$$\therefore SP.S'P = CD^2.$$

Example 1:

P and Q are extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S is a focus. Prove that $PQ^2 - (SP - SQ)^2 = 2b^2$.

Solution:

Let S be the focus on the positive side of the axis of x so that S is $(ae, 0)$.

Let P be $(a\cos\theta, b\sin\theta)$, then Q is $(-a\sin\theta, b\cos\theta)$.

Also, the focal distance $SP = a - ae\cos\theta$.

$$SQ = a - ae\cos(\theta + \frac{\pi}{2})$$

$$SQ = a + ae\sin\theta$$

Now, $PQ^2 - (SP - SQ)^2$

$$P = (a\cos\theta, b\sin\theta), Q = (-a\sin\theta, b\cos\theta)$$

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$PQ^2 = (a\cos\theta + a\sin\theta)^2 + (b\sin\theta - b\cos\theta)^2$$

$$PQ^2 - (SP - SQ)^2$$

$$= (a\cos\theta + a\sin\theta)^2 + (b\sin\theta - b\cos\theta)^2$$

$$-[a - ae\cos\theta - a - ae\sin\theta]^2$$

$$= a^2cos^2\theta + 2a\sin\theta\cos\theta + a^2sin^2\theta + b^2sin^2\theta - 2b\sin\theta\cos\theta + b^2cos^2\theta$$

$$-[(-ae\cos\theta) + (-ae\sin\theta)]^2$$

$$= a^2(cos^2\theta + sin^2\theta) + 2a^2\sin\theta\cos\theta + b^2(sin^2\theta + cos^2\theta) - 2b^2\sin\theta\cos\theta$$

$$-[a^2e^2sin^2\theta + 2a^2e^2\sin\theta\cos\theta + a^2e^2cos^2\theta].$$

$$= a^2 + 2a^2\sin\theta\cos\theta + b^2 - 2b^2\sin\theta\cos\theta - [a^2e^2 + 2a^2e^2\sin\theta\cos\theta]$$

$$= a^2 + b^2 - a^2e^2 + 2\sin\theta\cos\theta(a^2 - b^2 - a^2e^2)$$

$$= a^2(1 - e^2) + b^2 + 2\sin\theta\cos\theta(a^2e^2 - a^2e^2) \qquad (\text{Since } a^2 - b^2 = a^2e^2)$$

$= b^2 + b^2$ (Since $a^2(1-e^2) = b^2$

$= 2b^2.$

Example 2:

Prove that the acute angle between two conjugates diameters of an ellipse is a maximum when they are equal.

Solution:

Let CP, CD be two semi- conjugate diameters.

If P be $(a\cos\theta, b\sin\theta)$,

$$D \text{ is } \{a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\}$$

The slope of

$$CP = \frac{b\sin\theta}{a\cos\theta}$$

The slope of

$$CD = \frac{b\sin\left(\frac{\pi}{2}+\theta\right)}{a\cos\left(\frac{\pi}{2}+\theta\right)} = -\frac{b\cos\theta}{a\sin\theta}$$

The angle PCD= difference between the angles which the lines CP and CD make with the x-axis.

Let the angle PCD be α.

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\alpha = \frac{\frac{b\sin\theta}{a\cos\theta} + \frac{b\cos\theta}{a\sin\theta}}{1 - \frac{b\sin\theta}{a\cos\theta}\cdot\frac{b\cos\theta}{a\sin\theta}}$$

$$= \frac{ab\,(sin^2\theta + cos^2\theta)}{(a^2-b^2)\sin\theta\cos\theta} = \frac{ab}{(a^2-b^2)\frac{\sin 2\theta}{2}}$$

$$= \frac{2ab}{(a^2-b^2)\sin 2\theta}$$

∴ PCD will be a minimum when the variable part sin 2θ in the denominator is a maximum.

∴ $\sin 2\theta = 1$, which occurs when $2\theta = {}^{\pi}/_{2}$

$$(i.e.,)\ \theta = {}^{\pi}/_{4}$$

$$\therefore CP^2 = a^2cos^2\,{}^{\pi}/_{4} + b^2sin^2\,{}^{\pi}/_{4}$$

$$= {}^{1}/_{2}\,(a^2+b^2)$$

and $CD^2 = a^2sin^2\,{}^{\pi}/_{4} + b^2cos^2\,{}^{\pi}/_{4}$

$$= {}^{1}/_{2}\,(a^2+b^2)$$

$$\therefore CP = CD$$

∴ The minimum angle between the two conjugate diameters is then

$$\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right).$$

Example 3:

If P and Q are extremities of conjugate diameter of the ellipse, show that

(a) The locus of middle point of PD is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$

(b) The locus of the foot of the perpendicular on PD from the centre of the ellipse is $a^2x^2 + b^2y^2 = 2(x^2+y^2)$ and

(c) The locus of the point of intersection of the tangents at P and D is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

Solution:

Let $(a\cos\theta\,, b\sin\theta)$ be the co-ordinates of P, so that $(-a\sin\theta\,, b\cos\theta)$are the co-ordinates of D.

(a) If (x_1, y_1) is the middle point of PD.

$$x_1 = \frac{a}{2}(\cos\theta - \sin\theta) \text{ .... } (1)$$

$$y_1 = \frac{b}{2}(\sin\theta + \cos\theta) \text{ .... } (2)$$

The equation of the locus is found by eliminating θ from (1) and (2) .

From (1) , $\frac{2x_1}{a} = \cos\theta - \sin\theta$ ... (3)

From (2) , $\frac{2y_1}{b} = \sin\theta + \cos\theta$ ... (4)

Squaring (3) & (4) and adding , we get

$$\frac{4x_1^2}{a^2} = \cos^2\theta + sin^2\theta - 2\cos\theta\sin\theta$$

$$\frac{4y_1^2}{b^2} = \sin^2\theta + cos^2\theta + 2\cos\theta\sin\theta$$

$$\frac{4x_1^2}{a^2} + \frac{4y_1^2}{b^2} = 2(\cos^2\theta + sin^2\theta) - 2\cos\theta\sin\theta + 2\cos\theta\sin\theta$$

$$\div 4 \Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{1}{2}$$

∴ Locus of (x_1, y_1) is $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{1}{2}$.

(b) The eccentric angles of P and D are θ *and* $\pi/2 + \theta$

The equation of the line PD is

$$x/a \cos(45° + \theta) + y/b \sin(45° + \theta) = \cos 45°$$

The equation of the line perpendicular to the line PD passing through the centre is

$$\frac{x \sin(45° + \theta)}{b} - \frac{y}{b}\cos(45° + \theta) = 0$$

If (x_1, y_1) is the foot of the perpendicular from C to PD,

$$\frac{x_1}{a}\cos(45° + \theta) + \frac{y_1}{b}\sin(45° + \theta) = \cos 45° \ldots (1)$$

and

$$\frac{x_1}{b}\sin(45° + \theta) - \frac{y_1}{a}\cos(45° + \theta) = 0 \ldots (2)$$

From (2), $\sin(45° + \theta) = \frac{by_1}{ax_1} \cos(45° + \theta)$.

Substituting this in (1), we get,

$$x_1/a \cos(45° + \theta) + y_1/b \cdot by_1/ax_1 \cos(45° + \theta) = \cos 45°.$$

$$\cos(45° + \theta)\left\{x_1/a + y_1^2/ax_1\right\} = \frac{1}{\sqrt{2}}$$

$$\cos(45° + \theta) = \frac{ax_1}{\sqrt{2}(x_1^2 + y_1^2)} \quad \ldots . (3)$$

$$\sin(45° + \theta) = by_1/ax_1 \frac{ax_1}{\sqrt{2}(x_1^2 + y_1^2)}$$

$$\sin(45° + \theta) = \frac{by_1}{\sqrt{2}(x_1^2 + y_1^2)} \quad \ldots . (4)$$

Squaring (3) & (4) & adding, we get

$$\frac{a^2{x_1}^2}{2(x_1^2+y_1^2)^2}+\frac{b^2{y_1}^2}{2(x_1^2+y_1^2)^2}=1.$$

$$\Rightarrow\ a^2{x_1}^2+b^2{y_1}^2=2(x_1^2+y_1^2)^2$$

$\therefore$ Locus of (x_1, y_1) is

$$a^2x^2+b^2y^2=2(x^2+y^2)^2$$

(c) The tangent of P and D are

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1\ \ldots.(1)$$

$$\text{and } -\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta=1\ \ldots.(2)$$

Let the intersecting point of the two tangents by (x_1, y_1)

Then

$$\frac{x_1}{a}\cos\theta+\frac{y_1}{b}\sin\theta=1\ \ldots.(3)$$

$$-\frac{x_1}{b}\sin\theta+\frac{y_1}{b}\cos\theta=1\ \ldots.(4)$$

Eliminate θ from (3) & (4).

Squaring and adding (3) & (4) , we get;

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=2.$$

$\therefore$ Locus of (x_1, y_1) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2,$$

which is another similar and concentric ellipse.

1.5.2 Equi-conjugate diameters:

Definition:

When two conjugate diameters are equal, they are called equi- conjugate diameters.

Length of each equi-conjugate diameter = 2CP

$$= \sqrt{2(a^2 + b^2)}$$

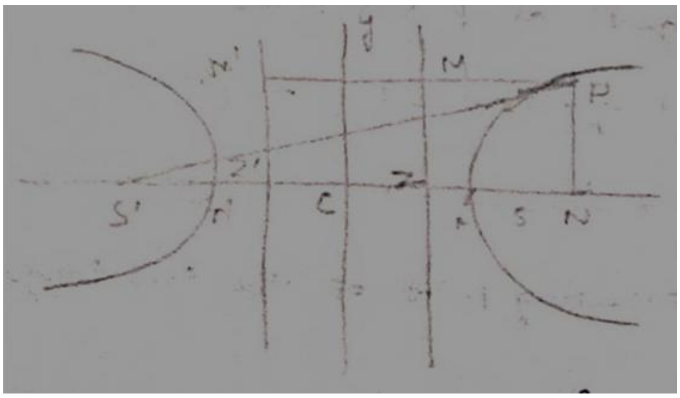
1.6 Conjugate diameters of hyperbola:

1.6.1 Hyperbola:

A hyperbola is the locus of a point which moves so that its distance from a fixed point (the focus) is $e(> 1)$ times its distance from a fixed straight line (the directrx)

1.6.2 Equation of hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



1.6.3 Conjugate Diameters:

$y = mx$ and $y = m_1x$ are conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $mm_1 = \frac{b^2}{a^2}$ and also that (1) chords parallel to the diameter will be bisected by the other diameter and (2) that the tangents at the extremities of one diameter are parallel to the other diameter.

1.6.4 Properties of Conjugate Diameters:

(1) If a pair of diameters be conjugate with respect to a hyperbola, they will be conjugate with respect to the conjugate hyperbola.

(2) If a diameter meets a hyperbola in real points, it will meet the conjugate Phyperbola in imaginary points; and the conjugate diameters will meet the conjugate hyperbola in real points.

$$\Rightarrow x = \pm a \tan\theta$$

$$y = \pm b \sec\theta.$$

$\therefore$ The conjugate diameter meets the conjugate hyperbola in the points D $(a\tan\theta, b\sec\theta)$, $D'(-a\tan\theta, -b\sin\theta)$ which is real.

(3) If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D, then

$$CP^2 - CD^2 = a^2 - b^2.$$

(4) The parallelogram formed by the tangents at the extremities of conjugate diameters has its vertices lying on the asymptotes and is of constant area.

$\therefore$ The area of the parallelogram $KLMN = 4ab$.

Unit-II:

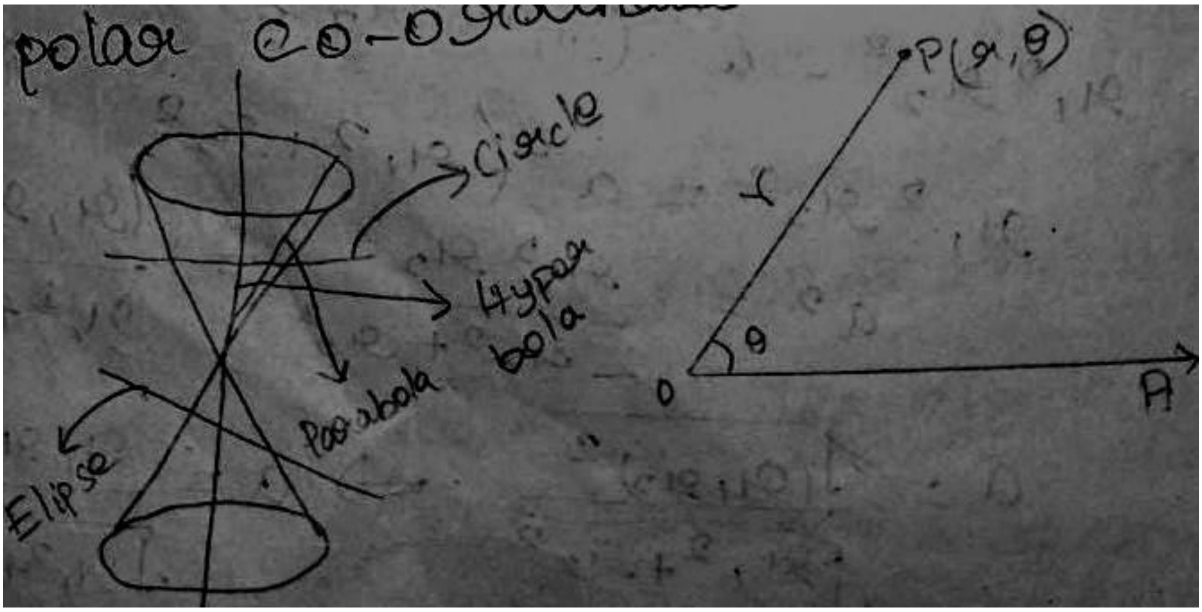
Polar coordinates: General polar equation of straight line – Polar equation of a circle given a diameter, Equation of a straight line, circle, conic – Equation of chord, tangent, normal.

2.1 Polar co-ordinates:

Still now we have discussed the system of Cartesian co-ordinates but there is another system of co-ordinates which is of frequents use in A.G.

Let OB→ fixed point, OA→ fixed line through O.

The position of a point P can be determined by distance r from 0 and θ angle of OAP. The point O is called the pole OA the initial line, r the radius vector and θ the vectorial angle and (r, θ) the polar co-ordinates of the point P.

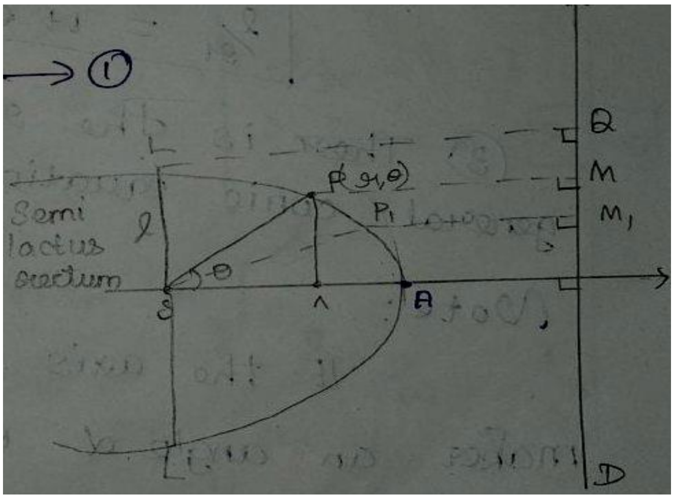


2.2 General polar equation of straight line:

2.2.1 Polar equation of the conic.

Let S be the focus and XM→ Direction of the conic. Ant . Let a small e be the eccentricity AS a initial SX and

Let P be a any point on the conic. It's co-ordinate be (r, θ), so that $SP = n$ and angle XSP be θ. Draw PM and PN perpendicular respectively to directrix and to the initial line.



Let LSL' be the lactuos rectum of the conic and $SL = l$.

$$e = \frac{SP}{PM} = \frac{r}{Nx} \quad \rightarrow (1)$$

$$e = \frac{SL}{LQ} = \frac{SL}{SX} = \frac{l}{SX} \quad \rightarrow (2)$$

$$SX = \frac{l}{e}.$$

$$NX = SX - SN$$

Here, in ΔPSN,

$$\cos\theta = \frac{SN}{SP} = \frac{SN}{r}$$

$$SN = r\cos\theta$$

$$NX = (SX - r\cos\theta)$$

From (1),

$$e = \frac{r}{NX}$$

$$e = \frac{r}{SX - r\cos\theta}$$

$$e = \frac{r}{\left(\frac{l}{e} - r\cos\theta\right)}$$

$$e\left(\frac{l}{e} - r\cos\theta\right) = r$$

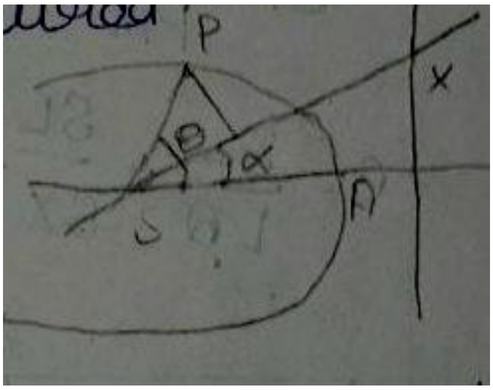
$$l - er\ \cos\theta = r$$

$$l = r + er\cos\theta$$

$$l = r(1 + e\cos\theta)$$

$$\frac{l}{r} = 1 + e\cos\theta \qquad \rightarrow (3)$$

Equation (3) is the required equation of general conic equation.



2.3. Polar equation of a circle given a diameter:

Note:

If the axis SX of the conic makes an angle α with the initial line SA then SP makes an angle $(\theta - \alpha)$ with initial line, so that the equation of the conic will be

$$\frac{l}{r} = 1 + e\,cos(\theta - \alpha)$$

2.4 Equation of a straight line, circle, conic:

2.4.1 Tracing the conic $\frac{l}{r} = 1 + e\,cos\theta$.

Case (i):

$$e = 0$$

$$\frac{l}{r} = 1 + 0 = 1$$

$\Rightarrow l = r \;\Rightarrow$ The conic will be circle.

Case (ii):

$$e = 1$$

$$\frac{l}{r} = 1 + cos\theta$$

Let θ = 0⇒ $\frac{l}{r} = 1 + 1$

$$\Rightarrow \frac{l}{r} = 2$$

$$r = \frac{l}{2}$$

If $\theta \uparrow_0^{\pi}$ then $r \uparrow_{l/2}^{\infty}$ and $\theta \uparrow_{\pi}^{2\pi}$ then $r \downarrow_{l/2}^{\infty}$

$\Rightarrow$ Then conic will be a parabola

Case (iii):

$0 < e < 1$

$\frac{l}{r} = 1 + e \cos \theta$

$r = \frac{l}{1+e \cos \theta}$

$0 = \pi \rightarrow r = \frac{l}{1-e}$ (maxi value of r)

$0 = 0 \rightarrow r = \frac{l}{1+e}$ (min value of r)

If $\theta \uparrow_{0}^{\pi}$ then $r \uparrow_{l/1+e}^{l/1-e}$

If $\theta \ \uparrow_{\pi}^{2\pi} \ then\, r \downarrow_{l/1-e}^{l/1+e}$

$\Rightarrow$ The conic will be ellipse.

Case (iv):

$e > 1$

$\frac{l}{r} = 1 + e \cos\theta$

$$r = \frac{l}{1 + e \cos\theta} = \frac{l}{1 - e}$$

$$\theta = 0 \Rightarrow r = \frac{l}{1 - e}$$

$$\theta = \cos^{-1}\left(-1/_{e}\right) \Rightarrow r = \infty$$

Then If $\theta \uparrow_{0}^{\cos^{-1}(-1/e)}$ then $r \uparrow_{l/1+e}^{\infty}$

If $\theta \ \uparrow_{\cos^{-1}(-1/e)}^{2\pi}$ then $r \downarrow_{\infty}^{l/1+e}$

The conic will be hyperbola.

Example 1:

Trace the curve $\frac{10}{r} = 3cos\theta + 4sin\theta + 5.$

Solution:

Given curve is $10/r = 3cos\theta + 4sin\theta + 5$

$\div\ by\ 5$

$$2/r = 1 + 3/5 \cos\theta + 4/5 \sin\theta$$

Let $\cos\alpha = 3/5 \ \Rightarrow \sin\alpha = \sqrt{1 - cos^2\alpha}$

$$= \sqrt{1 - 9/25}$$

$$= \sqrt{16/25} = 4/5$$

Hence $2/r = 1 + \cos\theta\cos\alpha + \sin\theta\sin\alpha$

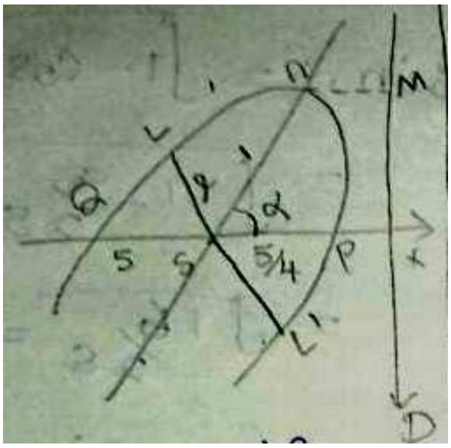
$$2/r = 1 + \cos(\theta - \alpha) \ \ when\ \alpha = \cos^{-1}\left(3/5\right)$$

$$\cong 53°r'$$

Hence the curve is

$$2/r = 1 + \cos(\theta - \alpha), \qquad \alpha \cong 53°r'$$

Here $l = 2, e = 1$ then, the curve is parabola with its focus at the pole initial point α is the angle between the initial line and axis $r = \frac{2}{1+\cos(\theta-\alpha)}$



$$\theta = 0 \Rightarrow r = \frac{2}{1+\cos(-\alpha)}$$

$$r = \frac{2}{1 + cos\alpha}$$

$$r = \frac{2}{1 + {}^{3}/_{5}} = {}^{10}/_{8} = {}^{5}/_{4}$$

$$\theta = \pi, \qquad r = \frac{2}{1+\cos(\pi-\alpha)}$$

$$= \frac{2}{1-\cos\alpha} = \frac{2}{1 - {}^{3}/_{5}}$$

$${}^{10}/_{2} = 5$$

If A is the vertex of the parabola, then $AS = 1$

If L and L' are the extremities of the lactus rectum then $SL' = 2$.

If the parabola meet at initial line at P its extension in the opposite direction at Q.

Then $SP = {}^{5}/_{4}$ $and\ SQ = 5$

From this we can get the shape of the curve.

Example 2:

Trace the curve ${}^{12}/_{r} = 4 + \sqrt{3}\cos\theta + 3\sin\theta$

Solution:

Given ${}^{12}/_{r} = 4 + \sqrt{3}\cos\theta + 3\sin\theta$

$\div\ by\ 4$

$$ {}^{3}/_{r} = 1 + {}^{\sqrt{3}}/_{4}\cos\theta + {}^{3}/_{4}\sin\theta $$

$$ {}^{3}/_{r} = 1 + {}^{\sqrt{3}}/_{4}\left({}^{1}/_{2}\cos\theta + {}^{\sqrt{3}}/_{2}\sin\theta\right) $$

$$ \cos\alpha = {}^{1}/_{2}, \qquad \sin\alpha = \sqrt{1 - cos^2\alpha} = \sqrt{1 - {}^{1}/_{4}} $$

$$ = {}^{\sqrt{3}}\Big/_{\sqrt{4}} = {}^{\sqrt{3}}/_{2} $$

$$ {}^{3}/_{r} = 1 + {}^{\sqrt{3}}/_{2}\ (\cos\alpha\cos\theta + \sin\alpha\sin\theta) $$

$$ {}^{3}/_{r} = 1 + {}^{\sqrt{3}}/_{2}\cos(\theta - \alpha) $$

$$ \alpha = \cos^{-1}({}^{1}/_{2}) = {}^{\pi}/_{3} $$

$$ {}^{3}/_{r} = 1 + {}^{\sqrt{3}}/_{2}\cos\left(\theta - {}^{\pi}/_{3}\right) $$

The equation of the conic is $l/_r = 1 + e\cos(\theta - \alpha)$

Here $l =$ semi lactus rectum$= 3$

$$e = \sqrt{3}/_2 \Rightarrow 0 < e < 1$$

Since $0 < e < 1$. The conic is the ellipse with focus at poles and the axis of the conic makes an angle $\pi/_3 = 60°$ with the initial line

$$l = \ b^2/_a \ \ and\ b^2 = \ a^2(1 - e^2)$$

$$l = \frac{a^2(1 - e^2)}{a} = a^2(1 - e^2)$$

$$3 = a\left(1 - 3/_4\right) = a/_4$$

$$a = 12\ since\ l = \ b^2/_a \ \Rightarrow 3 = 6^2/_{12}$$

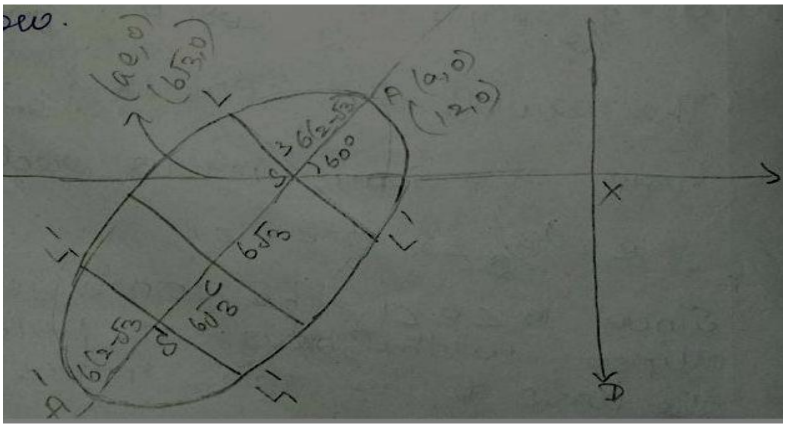
$$b^2 = 36 \ \Rightarrow b = 6$$

If another focus is S' then $SS' = \ 2ae = 2 \times 12\sqrt{3}/_2 = 12\sqrt{3}$

If A is the vertex near to S then $SA \ = \ CA - CS = \ 12 - 6\sqrt{3}$

$$= a - ae = \ 6(2 - \sqrt{3})$$

From this we can get the shape of the curve as below.



Example 3:

Trace the conic $^{2}/_{r} = 1 + \cos\theta + \sin\theta$.

Solution:

Given $^{2}/_{r} = 1 + \sqrt{2}\left(^{1}/_{\sqrt{2}}\cos\theta \ + {}^{1}/_{\sqrt{2}}\sin\theta\right)$

$$^{2}/_{r} = 1 + \sqrt{2}\,(\cos\alpha\cos\theta \ + \sin\alpha\sin\theta)$$

$$\text{where } \cos\alpha = {}^{1}/_{\sqrt{3}}, \quad \alpha = {}^{\pi}/_{4} = 45°$$

$$^{2}/_{r} = 1 + \sqrt{2}\,(\cos(\ \theta - {}^{\pi}/_{4})$$

W.K.T the equation of the conic is $^{l}/_{r} \ = 1 + e\cos(\theta - \alpha)$

Here $l =$ semi latus rectum $= 2$

$$\alpha = {}^{\pi}/_{2}\ , e = \sqrt{2}$$

Since $e > 1$, then the conic is the Hyperbola.

From the form of the equation, we get that one of the foci S of the conic is at the pole. The axis of the conic makes an angle $^{\pi}/_{4} = 45°$ with the initial lines. The semi lactus rectum l is 2. And the eccentricity of conic is $\sqrt{2}$, which is $e > 1$ then it represents a rectangular Hyperbola.

In the rectangular hyperbola, $a = b$

and $l = {b^2}/{a} \quad \Rightarrow l = a$

W.K.T. $l = 2$, then $a = 2$

If the other focus is S' then $SS' = 2ae = 2(2\sqrt{2}) = 4\sqrt{2}$

If the vertex near to S is A then SA=CS-CA $=2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

$$= ae - a$$

If the vertex near to S' is A' then $SA' = AS + AA' = 2(\sqrt{2} - 1) + 4$

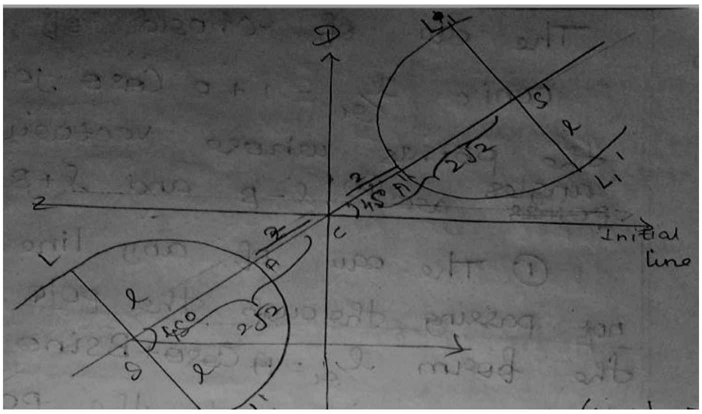
$$= 2\sqrt{2} + 2$$

So $1 + \sqrt{2}\cos(\theta - {\pi}/{4}) = 0 \; as$

$$\cos\left(\theta - {\pi}/{4}\right) = -{1}/{\sqrt{2}}$$

$$\cos\theta - {\pi}/{4} = \cos^{-1}\left({-1}/{\sqrt{2}}\right)$$

$$\theta = \cos^{-1}\left({-1}/{\sqrt{2}}\right) + {\pi}/{4}$$



$$\theta = \left(\pi + {\pi}/{4}\right) + {\pi}/{4} \; or \; \theta = \left(\pi - {\pi}/{4}\right) + {\pi}/{4}$$

Then $\theta = {}^{3\pi}/_{2}\ (or)\ \ \pi$

The line $\theta = \pi. and\ \theta = {}^{3\pi}/_{2}$ are parallel to the Asymptotes.

2.5 Equation of chord, tangent, normal:

2.5.1 The equation of chord of the conic $\boldsymbol{{}^{l}/_{r} = 1 + e\cos\theta}$ joining the points whose vectorial angle are $\boldsymbol{\alpha - \beta\ and\ \alpha + \beta}$.

1) The equation of any line not passing through the pole is of the form

${}^{l}/_{r} = A\cos\theta + \sin\theta$ where the vectorial angle the point P and Q is $\alpha - \beta$ and $\alpha + \beta$.

2) The vectorial angle of P becomes α then the equation of tangent at P become

$${}^{l}/_{r} = e\cos\theta\ + \cos(\theta - \alpha).$$

3) The equation of the tangent at α to the conic ${}^{l}/_{r} = 1 + e\cos(\theta - r)\ is$

$${}^{l}/_{r} = 1 + e\cos(\theta - r) + \cos(\theta - \alpha).$$

Example 1:

Find the condition in order that the line ${}^{l}/_{r} = A\cos\theta\ +\ B\sin\theta$ may be tangent to the conic ${}^{l}/_{r} = 1 + e\cos\theta$

Solution:

Here the conic is ${}^{l}/_{r} = 1 + e\cos\theta$ we have to find a condition that the line ${}^{l}/_{r} = A\cos\theta\ +\ B\sin\theta$ be the tangent to the conic ${}^{l}/_{r} = 1 + e\cos\theta$

The tangent at α to the conic $l/_{r} = 1 + e\cos\theta$.

$$l/_{r} = e\cos\theta \ + \cos(\theta - \alpha) \ \ [\therefore r = 0]$$

$$l/_{r} = e\cos\theta \ + \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$l/_{r} = (e\cos\theta\ )\cos\theta + \sin\alpha\sin\theta$$

$$\Rightarrow A = e + \cos\theta \quad \text{and} \quad B = \sin\alpha$$

$$A - e = \cos\alpha \quad \text{and} \quad B = sin\alpha$$

$$(A - e)^2 + B^2 = cos^2\alpha + sin^2\alpha$$

$$(A - e)^2 + B^2 = 1.$$

2.5.2 Asymptotes of the conic $l/_{r} = 1 + e\cos\theta$.

The tangent at α to this conic is $l/_{r} = e\cos\theta \ + \cos(\theta - \alpha)$

These tangents become an Asymptotes if the point of contact lies ∞.

That is the point whose V.A α lies on the conic at ∞ distance from the pole.

Asymptotes of the conic $l/_{r}$

$$r \to \infty \ l/_{r} \to 0$$

$$\Rightarrow 0 = 1 + e\cos\theta \ \Rightarrow \cos\alpha = {}^{-1}/_{e}$$

$$\sin\alpha = \pm\sqrt{1 - \left({}^{1}/_{e}\right)^2}$$

$$\sin\alpha = \ \pm\sqrt{e^2 - {}^{1}/_{e}}$$

$$l/_{r} = e\cos\theta \ + \cos(\theta - \alpha) \ \ [\therefore \alpha = 0]$$

$$l/_{r} = e\cos\theta \ + \cos\theta\cos\alpha + \sin\theta\sin\alpha$$

$$l/r = (e\cos\theta\ )\cos\theta + \sin\alpha\sin\theta$$

$$l/r = \left(e - \frac{1}{e}\right)\cos\theta \pm \frac{\sqrt{e^2-1}}{e}\sin\theta$$

$$l/r = \frac{e^2-1}{e}\cos\theta \pm \frac{\sqrt{e^2-1}}{e}\sin\theta$$

$$l/r = \frac{e^2-1}{e}\left(\cos\theta \pm \frac{1}{\sqrt{e^2-1}}\sin\theta\right),$$

which are the asymptotes of the conic.

2.5.3 Equation of the normal at the point P whose vertical angle is α.

The equation of the normal is $\frac{e\sin\alpha}{1+e\cos\alpha} \times \frac{l}{r} = e\sin\theta + \sin(\theta-\alpha)$

Example 1:

If the normal at L, one of the extremities of the conic $l/r = 1 + e\cos\theta$,meets the curve again in Q. Show that

$$SQ = l\left(\frac{1+3e^2-e^4}{1+e^2-e^4}\right)$$

Solution:

The co-ordinates of $L\ (l, \pi/2)$

The normal to the conic at α is $\frac{e\sin\alpha}{1+e\cos\alpha} \times \frac{l}{r} = e\sin\theta + \sin(\theta-\alpha)$.

At $L,\ \alpha = \pi/2$.

The normal at L is

$$\frac{e\sin {}^{\pi}/_{2}}{1+e\cos {}^{\pi}/_{2}}\times\frac{l}{r}=e\sin\theta+\sin\left(\theta-{}^{\pi}/_{2}\right)$$

$$\frac{el}{r}=e\sin\theta+\sin\theta\cos {}^{\pi}/_{2}-\sin {}^{\pi}/_{2}\cos\theta$$

$$ {}^{el}/_{r}=e\sin\theta\ -\cos\theta\quad\rightarrow(1)$$

The given conic equation is ${}^{l}/_{r}=1+e\cos\theta\quad\rightarrow(2)$

From (1) & (2)

$e(1+\cos\theta)\quad=e\sin\theta\ -\cos\theta$

$$\Rightarrow e+e^{2}\cos\theta+\cos\theta\ =e\sin\theta$$

$$\Rightarrow e+(1+e^{2})\cos\theta=e\sin\theta$$

$$(e+(1+e^{2})\cos\theta)^{2}=(e\sin\theta)^{2}$$

$$e^{2}+2e(1+e^{2})\cos\theta+(1+e^{2})cos^{2}\theta-e^{2}sin^{2}\theta=0$$

$$e^{2}(1-sin^{2}\theta)+2e(+e^{2})\cos\theta+(1+e^{2})cos^{2}\theta=0$$

$$\cos\theta\ \left(e^{2}\cos\theta+2e(1+e^{2})+(1+e^{2})cos\theta\right)=0$$

θ is not unique $\Rightarrow cos\ \theta\ \neq 0$

$\cos\theta(e^{2}+(1+e^{2})^{2})+2e(1+e^{2})=0$

$$\cos\theta\ =\frac{-2e\,(1+e^{2})}{e^{2}+(1+e^{2})^{2}}$$

Then the conic equation is

$$ {}^{l}/_{r}=1-\frac{2e^{2}(1+e^{2})}{e^{2}+(1+e^{2})^{2}}$$

Here r can take as SQ

$$ {}^{l}/_{SQ}=1-\frac{2e^{2}(1+e^{2})}{e^{2}+(1+e^{2})^{2}}$$

$$= \frac{1+e^2-e^4}{1+3e^2+e^4}$$

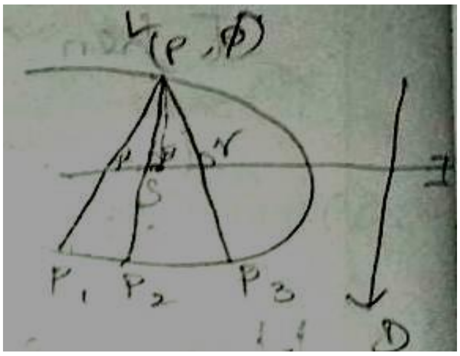
$$\therefore SQ = l.\frac{1+3e^2+e^4}{1+e^2-e^4}$$

Example 1:

If the normal at α, β, γ on $l/_r = 1 + e\cos\theta$ meets in the point (ρ, ∅), show that

(i) $\tan(\alpha/_2) + \tan(\beta/_2) + \tan(\gamma/_2) = 0$ $and\ (ii)\ \alpha + \beta + \gamma = 2n\pi + 2\emptyset$

Solution:



(i) The normal at θ_1 to the conic $l/_r = 1 + e\cos\theta$

$$\frac{sin\theta_1}{1+\theta_1}\ l/_r = \sin\theta + \sin(\theta - \theta_1)$$

Here this normal meet at $(\rho, \emptyset)$,

$$\frac{sin\theta_1}{1+cos\theta_1}\ l/_\rho = \sin\emptyset + \sin(\emptyset - \theta_1)$$

$$\frac{sin\theta_1}{1+cos\theta_1}\ l/_\rho = \sin\emptyset + \sin\emptyset\cos\theta_1 - \cos\emptyset\sin\theta_1$$

$$\left[\because \sin 2\theta = \frac{2\ \tan\theta}{1 + tan^2\theta}\right]$$

Now,

Consider tan ${}^{\theta_1}/_2 = t$

$$sin\theta_1 = {}^{2t}/_{1+t^2}$$

$$cos\theta_1 = {}^{1-t^2}/_{1+t^2}$$

$$\frac{{}^{2t}/_{1+t^2}}{1 + {}^{1-t^2}/_{1+t^2}}\ {}^{l}/_{\rho} = \sin\emptyset + \sin\emptyset\,{}^{1-t^2}/_{1+t^2} - \cos\emptyset\,{}^{2t}/_{1+t^2}$$

$$(1+t^2)\,{}^{l}/_{\rho} = (1+t^2)\sin\emptyset + \sin\emptyset\,(1-t^2) - \cos\emptyset\,.\,2t$$

$$lt^3 + lt = 2\rho\sin\emptyset + \emptyset e\,t^2\ \sin\emptyset + \rho\sin\emptyset - \rho\sin\emptyset t^2 - \rho cos\emptyset 2t.$$

$$lt^3 + (2\rho cos\emptyset + l)l - 2\rho\sin\emptyset = 0$$

$$t^3 + \left(\frac{2\rho cos\emptyset + l}{l}\right)t - \left(\frac{2\rho\sin\emptyset}{l}\right) = 0$$

[**Note:**

$$t^3 + at^2 + bt + c = 0$$

Three roots t_1, t_2, t_3

$$t_1 + t_2 + t_3 = -a$$

$$t_1t_2 + t_2\,t_2 + t_1\,t_3 = b$$

$$t_1t_2t_3 = -c]$$

Here $a = 0$

$$b = \frac{2\rho cos\emptyset + l}{\rho}$$

$$c = \frac{\rho sin\emptyset}{l}$$

Corresponding to these values t, to these values t, let the value of θ_1 be α, β, γ

$$\Rightarrow \tan\frac{\alpha}{2} = t_1, \tan\frac{\beta}{2} = t_2\, , \tan\frac{\gamma}{2} = t_3$$

Here

$$t_1 + t_2 + t_3 = 0$$

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = 0$$

(ii)

$$\tan(a + b + c) = \frac{(\, tana + tanb + \tan c) - (\, tana + tanb + \tan c)}{1 - (\, tanatan\, b + tanbtanc + \tan c\, tana)}$$

Let

$$a = \frac{\alpha}{2} \ \ , b = \frac{\beta}{2}, c = \frac{\gamma}{2}$$

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2}\right) = \frac{0 - \frac{\rho sin\emptyset}{l}}{1 - \frac{2\rho cos\theta + l}{l}}$$

$$= \tan\emptyset$$

$$\tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2}\right) = \tan\emptyset$$

$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \emptyset + n\pi$$

$$\alpha + \beta + \gamma = 2(\emptyset + n\pi)$$

$$\alpha + \beta + \gamma = 2n\pi + 2\emptyset.$$

Unit-III:

System of Planes-Length of the perpendicular–Orthogonal projection.

3.1 System of Planes:

The equation which is satisfied by the coordinates of any point in the is called the equation of the plane.

3.1.1 The general equation of the first degree in x,y,z represents a plane.

The general equation of the first degree in x,y,z is

$Ax+By+Cz+D=0 \rightarrow (1)$

If $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ be any two points on the locus,

We have, $Ax_1+By_1+Cz_1+D=0 \rightarrow (2)$

$Ax_2+By_2+Cz_2+D_2=0 \rightarrow (3)$

$\times$ eqn (3) by k and adding it with 2, we get

$A(x_1+kx_2)+B(y_1+ky_2)+C(z_1+kz_2)+D(1+k)=0$

$\div(1+k)$

$$A.\frac{x_1+kx_2}{1+k} + B\frac{y_1+ky_2}{1+k} + C\frac{z_1+kz_2}{1+k} + D = 0$$

This shows that the point $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$ lies on the locus $Ax+By+Cz+D=0$

The point $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$ divides the line joining the points (x_1,y_1,z_1) (x_2,y_2,z_2) in the ratio k:1. As k can have any value positive or negative, it follows that the all the points in the straight line p,q satisfies the eqn(1) i.e. The line p,q lines completely in the locus p and q any two points on the locus.

Hence we get that if we join any two points on the locus, represented by eqn(1) that line now far if it is produced in both sides lies completely on the locus.

∴ The given eqn represent a plane. Hence, we get that every eqn of the first degree in x, y, z. represents a plane.

3.1.2 The equation of the plane intercept a,b,c on the axis Ox, Oy, Oz.

Let the given plane meet the coordinate axis Ox, Oy, Oz at A, B, C

∴ OA=a, OB=b, OC=c, hence the coordinates of the points A,B,C are respectively (a,0,0), (0,b,0), (0,0,c). Let the eqn of the plane by

Px+Qy+Rz+S=0→(1)

Sub points in (1),

Pa+S=0 ⇒ P=-s/a

Qb+S=0 ⇒ Q=-s/b

Rc+S=0 ⇒ R=-s/c

$$-\frac{s}{a}x-\frac{s}{b}y-\frac{s}{c}z+s=0$$

Sub these values P,Q,R, we get $-\frac{s}{a}x-\frac{s}{b}y-\frac{s}{c}z+s=0$

$(-)(\div S)$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

This is known as intercept form of the equation of a plane .

3.1.3 The equation on the plane in terms of P, the length of the perpendicular from the origin to it l, m, n; the direction cosines of the perpendicular.

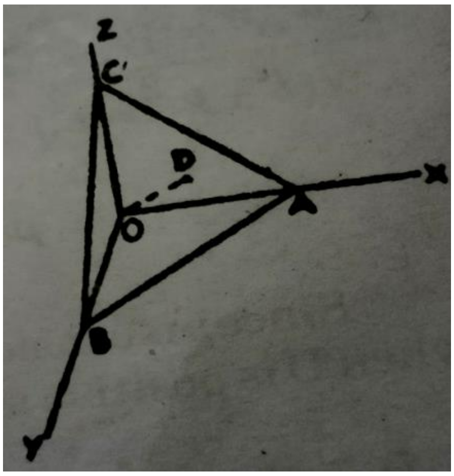
Then the given plane makes intercept a, b, c and Ox, Oy, Oz.

Let the perpendicular OD from O the plane ABC be plane (P). OD is perpendicular to any line the plane ABC

$\because \angle ADO = 90^{\cdot}$

$\because \cos(\angle DOA) = l$

Since the direction cosines of the perpendicular OD are (l, m, n).



In triangle ODA,

$$\cos \angle DOA = \frac{OD}{OA}$$

$$l = \frac{p}{a}$$

$$\therefore a = \frac{p}{l}$$

|||ly,

$$b = \frac{p}{m}, c = \frac{p}{n}$$

The eqn of the plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{zc}=1$

Substitute these values a, b, c, the eqn of the plane becomes $\frac{lx}{p}+\frac{my}{p}+\frac{nz}{p}=1$

(i.e) $lx+my+nz=p$ (normal form)

This is known as the normal form of the equation of the plane.

Note:

The perpendicular form the origin on any plane always taken to be positive.

3.1.4 We have now obtained several forms for the equations of a plane.

1. $Ax+By+Cz+D=0 \rightarrow (1)$
2. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \rightarrow (2)$
3. $\mathrm{lx}+\mathrm{my}+\mathrm{nz}=\mathrm{p} \rightarrow (3)$

we can easily see that all the above forms are equivalent.

Equation (1) can be expressed in the form (3).

Since these two equations represent the same pane,

we have (1)÷(3)

$$\frac{A}{l}=\frac{B}{m}=\frac{C}{n}=-\frac{D}{p}.$$

Hence each ratio is equal to $\pm\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{l^2+m^2+n^2}}= \pm\sqrt{A^2+B^2+c^2}$

Since $l^2+m^2+n^2$=1

P=$\pm\frac{D}{\sqrt{A^2+B^2+C^2}}$

Since p is considered to be always positive, take the positive sign when D is positive and the negative sign when D is negative.

$$l = \mp\frac{A}{\sqrt{A^2+B^2+C^2}},\ m = \mp\frac{B}{\sqrt{A^2+B^2+C^2}},\ n = \mp\frac{c}{\sqrt{A^2+B^2+C^2}}$$

The negative sign to be used when D is positive and the positive sign when D is negative

The general equation can be in the form

$$\frac{x}{-\frac{D}{A}}+\frac{y}{-\frac{D}{B}}+\frac{z}{-\frac{D}{C}}=1$$

Hence the intercepts on the coordinate axes are $\frac{-D}{A}, \frac{-D}{B}\ and\ \frac{-D}{C}$

The general equation of a plane $Ax + By + Cz = 0$ contains three arbitrary constants, since we can write down the equation in the form,

$$\frac{A}{D}x+\frac{B}{D}y+\frac{C}{D}z+1=0$$

i.e. $px + qy + rz + 1 = 0$

Hence a plane has three degrees of freedom. So a plane can be drawn to satisfy three conditions such as:

1) passing through three non-collinear points
2) passing through two given points and perpendicular to a given planes; and
3) passing through a given point and perpendicular to two given planes.

3.1.5 The equation of the plane passing through the points

(x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3)

Let the equation of the plane be, $Ax + By + Cz + D = 0 \rightarrow (1)$

Since the plane passes through the points (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3) we get

$$Ax_1 + By_1 + Cz_1 + D = 0 \qquad \rightarrow (2)$$

$$Ax_2 + By_2 + Cz_2 + D = 0 \qquad \rightarrow (3)$$

$$Ax_3 + By_3 + Cz_3 + D = 0 \qquad \rightarrow (4)$$

Eliminating A, B, C, D from (1), (2), (3), (4), we get

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0,$$

which is the equation of the plane.

3.1.6 Direction cosines of the line which is perpendicular to a plane.

Let the equation of the plane be

$$Ax + By + Cz + D = 0 \rightarrow (1)$$

Let the direction ratios of any line perpendicular to the plane through the origin is $kl, km, kn,$ where k is some constant.

Hence the equation of the plane is of the form

$$klx + kmy + knz = p \quad \rightarrow (2)$$

Equation (1) and (2) represent the same plane.

$$\therefore \ \frac{A}{kl} = \frac{B}{km} = \frac{C}{kn}$$

$\therefore$ l, m, n are proportional to A, B, C.

3.1.7. Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

Angle between two planes is equal to the angle between the normal to them. The direction cosines of the normal to the two give panes are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively.

Hence the actual direction cosines or the normal are respectively,

$$\pm\frac{a_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \pm\frac{b_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \pm\frac{c_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

$$\pm\frac{a_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \pm\frac{b_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \pm\frac{c_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

If θ be the angle between the two planes, then

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

Cor.1.

If the planes be at right angles

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Cor.2.

If the planes are parallel then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Cor.3.

If the equation of two planes differs only in the constant term, they are parallel.

Thus, the planes $ax + by + cz + d = 0$ and$ax + by + cz + d_1 = 0$ are parallel.

Example 1:

Find the equation to the plane through $(3, 4, 5)$ parallel to the plane

$2x + 3y - z + k = 0$.

Solution:

The equation to any plane parallel to this plane is $2x + 3y - z + k = 0$.

If it passes through the point (3,4,5).

$$2(3) + 3(4) - 5 + k = 0$$

i.e., $k = -13$.

Hence the equation of the required plane is $2x + 3y - z - 13 = 0$.

Example 2:

Find the angle between the planes$2x - y + z = 6\,, x + y + 2z = 3$.

Solution:

The direction cosines of the normal to the planes are proportional to $2, -1, 1$ and $1, 1, 2$ respectively.

If θbe the angle between the planes, then

$$\cos\theta = \frac{2 - 1 + 2}{\sqrt{(2^2 + (-1)^2 + 1^2) \cdot \sqrt{(1^2 + 1^2 + 2^2)}}} = \frac{3}{\sqrt{6} \cdot \sqrt{6}} = \frac{1}{2}$$

$$\therefore\ \theta = \frac{\pi}{3}.$$

Example 3:

Find the distance of the origin from the plane $6x - 3y + 2z - 14 = 0$.

Solution:

Let the equation of the plane in the normal form be $lx + my + nz = p$.

$$\therefore \frac{l}{6} = \frac{m}{-3} = \frac{n}{2} = \frac{p}{14}$$

i.e.,

$$\frac{p}{14} = \frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(6^2+(-3)^2+2^2)}} = \frac{1}{7}$$

$$\therefore p = 2.$$

Example 4:

Find the equation of the plane passing through the points $(3,1,2)$, $(3,4,4)$ and perpendicular to the plane $5x + y + 4z = 0$.

Solution:

Let the equation of the required plane be $Ax + By + Cz + D = 0$.

Since this plane passes through the points $(3,1,2) and (3,4,4)$ we get

$$3A + B + 2C + D = 0 \qquad \rightarrow (1)$$

$$3A + 4B + 4C + D = 0 \qquad \rightarrow (2)$$

The plane is perpendicular to the plane $5x + y + 4z = 0$.

$$\therefore \quad 5A + B + 4C = 0 \qquad \rightarrow (3)$$

Solving the equation (1), (2) and (3), we get

$$A = -D; B = -D; c = \frac{3D}{2}$$

Substituting these values in the equation of the plane, we get

$$\text{-Dx} - Dy + \frac{3D}{2}z + D = 0$$

i.e., $2x + 2y - 3z - 2 = 0.$

Example 5:

Find the equation of the plane which passes through the point $(-1,3,2)$ and perpendicular to the two planes $x + 2y + 2z = 5\,, 3x + 3y + 2z = 8.$

Solution:

Let the equation of the required plane b $Ax + By + Cz + D = 0.$

It passes through the point $(-1,3,2)$.

$$\therefore \quad -A + 3B + 2C + D = 0 \quad \ldots (1)$$

The plane is perpendicular to the panes

$x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$

$$\therefore \quad A + 2B + 2C = 0 \qquad \ldots . (2)$$

$$3A + 3B + 2C = 0 \qquad \ldots (3)$$

From the equation (2) and (3), we get

$$\frac{A}{-2} = \frac{B}{4} = \frac{C}{-3}$$

Let each be equal to k

Then $A = -2k, B = 4k, C = -3k.$

Substituting he values of A, B, C in equation (1), we get D=-8k.

Hence the equation f the plane is

$$-2kx + 4ky - 3kz - 8k = 0 \;\; i.e., 2x - 4y + 3z + 8 = 0.$$

Example 6:

Find the equation of the plane passing through the points $(2, -5, -3), (-2, -3, 5)$ and $(5, 3, -3)$.

Solution:

Let the equation of the plane be $Ax + By + Cz + D = 0$.

Since it passes through $(2, -5, -3)$, we get

$$2A - 5B - 3C + D = 0 \qquad \ldots (1)$$

Similarly, $$-2A - 3B + 5C + D = 0 \qquad \ldots . (2)$$

$$5A + 3B - 3C + D = 0 \qquad \ldots \ldots (3)$$

Subtracting (2) from (1), we get

$$4A + 8B - 8C = 0$$

i.e.,$A + 2B - 2C = 0 \qquad \ldots \ldots (4)$

subtracting (3) from (2) we get

$$7A + 6B - 8C = 0 \qquad \ldots . (5)$$

From (4) and (5), we get

$$\frac{A}{-4}=\frac{B}{-6}=\frac{C}{-8}$$

i.e.,

$$\frac{A}{2}=\frac{B}{3}=\frac{C}{4}$$

Let each be equal to k.

Then $A = 2k, B = 3k, C = 4k$.

Substituting the value of A, B, C in (1), we get D=$-7k$.

Hence the equation of the required plane is

$2kx + 3ky + 4kz - 7k = 0$ i.e., $2x + 3y + 4z - 7 = 0$.

3.1.8. The ratio in which the plane $\boldsymbol{ax + by + cz + d = 0}$ divides the line joining the points $(\boldsymbol{x_1, y_1, z_1}), (\boldsymbol{x_{2,} y_2, z_2})$.

Let the required ratio be $\lambda : 1$.

Then the coordinates of the point which divides the straight line joining the points $(x_1, y_1, z_1) and\ (x_{2,} y_2, z_2)$ is

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$$

This point lies in the plane $ax + by + cz + d = 0$.

$$\therefore a\left(\frac{x_1+\lambda x_2}{1+\lambda}\right)+b\left(\frac{y_1+\lambda y_2}{1+\lambda}\right)+c\left(\frac{z_1+\lambda z_2}{1+\lambda}\right)+ d = 0$$

i.e.,

$$(ax_1 + by_1 + cz_1 + d) + \lambda(ax_2 + by_2 + cz_2 + d) = 0$$

$$\therefore\ \lambda = -\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}.$$

(8) This result gives us a method to determine whether two points are in the same side or on the opposite sides of plane.

When $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ have opposite signs, λ is positive. In that case $(x_1, y_1, z_1) and \left(x_{2,} y_2, z_2\right)$ lie on opposite sides of the plane $ax + by + cz + d = 0$.

When $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ have the same sign, λ is negative. In case $(x_1, y_1, z_1) and \left(x_{2,} y_2, z_2\right)$ lie on the same side of the plane.

Thus, we get that the two points $(x_1, y_1, z_1) and \left(x_{2,} y_2, z_2\right)$ lie on the same or different sides of the plane $ax + by + cz + d = 0$, according as the expressions $ax_1 + by_1 + cz_1 + d, ax_2 + by_2 + cz_2 + d$ are of the same or different signs.

3.1.9. Equation of a plane through the line of intersection of two given planes.

Let $ax_1 + by_1 + cz_1 + d = 0 \quad \ldots (1)$ and $ax_2 + by_2 + cz_2 + d = 0 \quad \ldots\ldots (2)$ be the equations of the two given planes.

Consider the equation

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0 \ \ldots (3)$$

Where the λ is any constant.

If is an equation of the first degree and therefore represents a plane.

Let (x_1, y_1, z_1) be a point on the intersecting line of the two planes (1) and (2).

$\therefore\ a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$ and $a_2x_2 + b_2y_2 + c_2z_2 + d_2 = 0$.

$\therefore\ a_1x_1 + b_1y_1 + c_1z_1 + d_1 + \lambda(\ a_2x_2 + b_2y_2 + c_2z_2 + d_2) = 0$

Hence values of (x_1, y_1, z_1) satisfy the equation (3) also.

∴ The plane represented by (3) passes through the points common to (1)and (2), i.e., passes through the intersecting line of (1) and (2). Since λ is any arbitrary constant (3) represents any plane through the intersecting lines of (1) and (2).

Conversely if the equation of any plane can be put in the form

$a_1x_1 + b_1y_1 + c_1z_1 + d_1 + \lambda(a_2x_2 + b_2y_2 + c_2z_2 + d_2) = 0$ where λ is any arbitrary constant, we conclude that it always passes through a fixed line, viz., the line of intersection of the planes

$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$ and $a_2x_2 + b_2y_2 + c_2z_2 + d_2 = 0$

Example 1:

Find the equation of the plane through the point $(1, -2,3)$and the intersection of the plane $2x - y + 4z = 7$ and $x + 2y - 3z + 8 = 0$.

Solution:

The plane

$$2x - y + 4z - 7 + \lambda(x + 2y - 3z + 8) = 0 \qquad \ldots.(1)$$

Passes through the intersection of the given planes for all values of λ.

It passes through the point $(1, -2,3)$.

$$\therefore 2 + 2 + 12 - 7 + \lambda(1 - 4 - 9 + 8) = 0.$$

i.e., $9 - 4\lambda = 0$

$$i.e., \lambda = \frac{9}{4}$$

substituting $\lambda = \frac{9}{4}$ in (1), we get

$$2x - y + 4z - 7 + \frac{9}{4}(x + 2y - 3z + 8) = 0\,,$$

$$i.e., 17x + 14y - 11z + 44 = 0.$$

Example 2:

Find the equation of the plane through the line of intersection of the planes

$x + y + z = 1, 2x + 3y + 4z - 7 = 0$ and perpendicular to the plane

$x - 5y + 3z = 5.$

Solution:

The equation of the required plane is of the form

$$x + y + z - 1 + \lambda(2x + 3y + 4z - 7) = 0,$$

$$i.e., (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - (1 + 7\lambda) = 0$$

This plane is perpendicular to $x - 5y + 3z = 5.$

$$\therefore\ (1 + 2\lambda) + (1 + 3\lambda)(-5) + (1 + 4\lambda)3 = 0$$

$i.e., -\lambda - 1 = 0\ \ i.e., \lambda = -1.$

Hence the required plane is

$$x + y + z - 1 - (2x + 3y + 4z - 7) = 0$$

$$i.e., -x - 2y - 3z - 6 = 0\ \ i.e., x + 2y + 3z + 6 = 0.$$

3.2. Length of the perpendicular.

We have seen that the length of the perpendicular form the origin to the plane

$$ax + by + cz += 0\ , is\ \ \pm\frac{d}{\sqrt{(a^2 + b^2 + c^2)}}$$

the positive sign is to be taken when d is positive and the negative sign when d is negative .

3.2.1. The length of the perpendicular from the point $(\boldsymbol{x_1, y_1, z_1})$ on the plane $\boldsymbol{lx + my + nz = p}$.

Consider a plane through (x_1, y_1, z_1) parallel to the given plane $lx + my + nz = p$.

Since it has the same normal as the given plane, its equation will be

$lx + my + nz = p_1$, where p_1 is the length of the perpendicular form the origin to the plane.

Since the point (x_1, y_1, z_1) is a point o this plane,

$$lx_1 + my_1 + nz_1 = p_1.$$

We can easily see that the length of the perpendicular form the point (x_1, y_1, z_1) on the given plane is $= p - p_1$.

$$i.e., = p - lx_1 - my_1 - nz_1.$$

3.2.2. The length of the perpendicular from the point $(\boldsymbol{x_1, y_1, z_1})$ on the plane $\boldsymbol{ax + by + cz + d = 0}$.

Let the normal form of the equation of the plane

$ax + by + cz + d = 0$ be $lx + my + nz = p$.

$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{m}{c} = -\frac{p}{d} \pm \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{1}{\sqrt{(a^2 + b^2 + c^2)}}$$

$$\therefore l = \pm \frac{a}{\sqrt{(a^2 + b^2 + c^2)}} \,; m = \pm \frac{b}{\sqrt{(a^2 + b^2 + c^2)}}$$

$$n = \pm \frac{c}{\sqrt{(a^2 + b^2 + c^2)}} \; and \; p = \mp \frac{d}{\sqrt{(a^2 + b^2 + c^2)}}$$

Now the length of the perpendicular by the previous article is

$$p - lx_1 - my_1 - nz_1.$$

Substituting the value of p, l, m, n in this expression ,we get the length of the perpendicular.

$$= \mp \frac{d}{\sqrt{(a^2 + b^2 + c^2)}}$$

$$- \left\{ \pm \frac{ax_1}{\sqrt{(a^2 + b^2 + c^2)}} \pm \frac{by_1}{\sqrt{(a^2 + b^2 + c^2)}} \pm \frac{cz_1}{\sqrt{(a^2 + b^2 + c^2)}} \right\}$$

$$= \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}}$$

Since the perpendicular form the origin on any plane have to be positive, the positive sign will be taken when d is positive and the negative sign when d is negative.

3.3 Orthogonal Projection:

3.3.1. The equation of the planes bisecting the angle between the plane

$\boldsymbol{a_1x + b_1y + c_1z + d_1 = 0}$ **and** $\boldsymbol{a_2x + b_2y + c_2z + d_2 = 0}$.

From any point (x, y, z) on the required plane the perpendicular on the two given planes will be equal. Hence the required equation is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

The $\pm$ sign gives the two different planes bisecting the angle between the two given planes.

Note 1:

The bisector of the acute angle makes with either of the planes of an angle which is less than 45° and the bisector of the obtuse angle makes with either of them an angle which is greater than 45°. This gives us a method to find the equation of the plane bisecting the acute or the obtuse angle between the given planes.

Note 2: Write down the equations of the planes so that d_1 and d_2 are positive.

Consider the equation

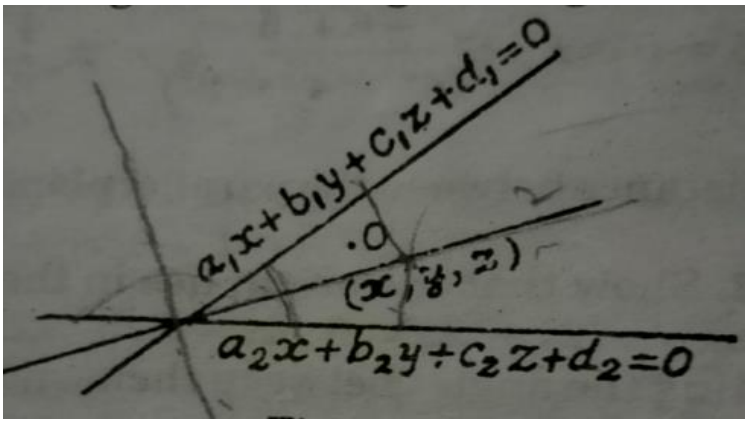
$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}} \quad \dots (1)$$

Since both denominators are positive, the numerators are both positive or both negative.

Case (i):

Let $a_1x + b_1y + c_1z + d_1$ and $a_2x + b_2y + c_2z + d_2$ be both positive.

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.



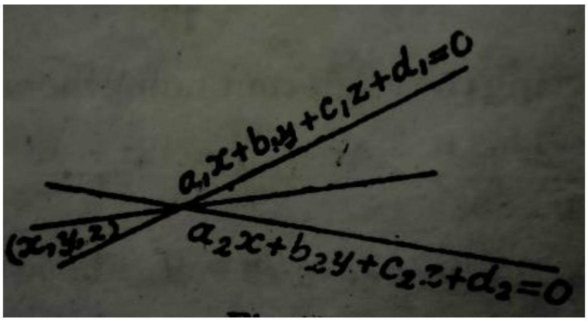
$i.e.$, the pint (x, y, z) satisfying the equation (1) and the origin lie on the same side of both the planes. This will be the case when (1) is bisector of the angle containing the origin.

Case (ii):

Let $a_1x + b_1y + c_1z + d_1$ and $a_2x + b_2y + c_2z + d_2$ be both negative.

Then the origin and (x, y, z)satisfying the equation (1) lie on the opposite sides of the planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.



This will be the case when (1) is the bisector of the angle containing the origin. In both the cases (1) is the bisector of the angle containing the origin.

Similarly,

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

Represents the plane bisecting the other angle between the given planes.

Example 1:

Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and

$4x - 4y + 2z + 5 = 0.$

Find a point on the plane $2x - 2y + z + 3 = 0$ and the distance between the two parallel planes is the perpendicular distance from that the point to the plane $4x - 4y + 2z + 5 = 0$.

Solution:

The first plane meets the z-axis at the point $(0,0,-3)$.

The length of the perpendicular from $(0,0,-3)$ to the plane

$4x - 4y + 2z + 5 = 0$ is

$$\pm \frac{-6+5}{\sqrt{(4^2+4^2+2^2)}} = \pm \frac{1}{6}.$$

Hence the distance between the parallel is

$$\frac{1}{6}.$$

Example 2:

Show that the origin lies in the acute angle between the planes

$x + 2y + 2z = 0, 4x - 3y + 12z + 13 = 0$. Find the planes bisecting the angle between them and point out which bisects the obtuse angle.

Solution:

The equation of the planes which bisects the angles between the given planes are given by

$$\frac{x+2y+2z-9}{\sqrt{(1^2+2^2+2^2)}} = \pm\frac{4x-3y+12z+13}{\sqrt{(4^2+3^2+12^2)}}$$

$$i.e., \frac{x+2y+2z-9}{3} = \pm\frac{4x-3y+12z+13}{13}.$$

The plane bisecting the angle containing the origin

$$\frac{-x-2y-2z+9}{3} = \frac{4x-3y+12z+13}{13}$$

$$i.e., 25x+17y+62z-78=0$$

Hence the plane bisecting the other angle is

$$\frac{4x-3y+12z+13}{13} = \frac{x+2y+2z-9}{3}$$

$$i.e., x+35y-10z-156=0.$$

Let us find the angle θ between

$$25x+17y+62z-78=0 \;\; and \; x+2y+2z=9.$$

$$\cos\theta = \frac{25+34+124}{\sqrt{(1^2+2^2+2^2)}\cdot\sqrt{(25^2+17^2+62^2)}} = \frac{61}{\sqrt{4758}}$$

$$tan^2\theta = sec^2\theta - 1$$

$$= \frac{4578}{61^2} - 1 = \frac{1037}{(61)^2}$$

$\therefore tan\theta = \frac{\sqrt{1037}}{61}$ which is less than 1. $\therefore$ $\theta < 45°$

$\therefore 25x+17y+62z-78=0$ is the angle containing the obtuse angle is

$$x+35y-10z-156=0.$$

Example 3:

Prove that the reflection of the plane

$ax + by + cz = d = 0$ in the plane $a_1x + b_1y + c_1z + d_1 = 0$ is the plane.

Solution:

$2(aa_1 + bb_1 + cc_1)(a_1x + b_1y + c_1z + d_1)$

$$= (a_1^2 + b_1^2 + c_1^2)(ax + by + cz + d).$$

The reflection of the plane

$$ax + by + cz + d = 0 \quad \dots (1)$$

in the plane $a_1x + b_1y + c_1z + d_1 = 0 \quad \dots (2)$

is the plane passing through the intersection of (1) and (2).

Its equation is of the form

$$ax + by + cz + d + \lambda(a_1x + b_1y + c_1z + d_1) = 0 \quad \dots (3)$$

Also, the perpendicular from any point on (2) to (1) and (3) are equal.

Let (x_1, y_1, z_1) be any point in (2).

Then $a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$

Then

$$\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}}$$

$$= \frac{ax_1 + by_1 + cz_1 + d + \lambda(a_1x_1 + b_1y_1 + c_1z_1 + d_1)}{\sqrt{\{(a + \lambda a_1)^2 + (b + \lambda b_1)^2 + (c + \lambda c_1)^2\}}}$$

$$\therefore \; a^2 + b^2 + c^2 = (a + \lambda a_1)^2 + (b + \lambda b_1)^2 + (c + \lambda c_1)^2$$

$$= a^2 + b^2 + c^2 + 2\lambda(aa_1 + bb_1 + cc_1) + \lambda^2(a_1^2 + b_1^2 + c_1^2)$$

$$i.e., \lambda(a_1^2 + b_1^2 + c_1^2) + 2(aa_1 + bb_1 + cc_1) = 0.$$

$$i.e., \lambda = -\frac{2(aa_1 + bb_1 + cc_1)}{a_1^2 + b_1^2 + c_1^2}$$

Substituting this value of λ in (3), we get the required result.

Exercises:

1.Find the equation of the plane through (1,1,1) and the line of intersection of the planes $x + 2y - z + 1 = 0, 3x - y + 4z + 3 = 0.$

2. Find the equation of the plane through the origin and the line of intersection of the planes $3x - y + 2z = 0, x + y + z = 1.$

3. Find the equation of the plane passing through the line of intersection of the planes $2x - 2y + 5z - 3 = 0$ and $4x + 2y - z + 7 = 0$ and parallel to the z-axis.

4. Find the equation of the plane passing through the line of intersection of the planes $2x + y + 3z - 4 = 0$ and $4x - y + 5z - 7 = 0$ and which is perpendicular to the plane $x + 3y - 4z + 6 = 0.$

6. Find the equation of the plane passing through the line of intersection of the planes $2x - 2y - z + 3 = 0$ and $3x + 5y - 2z - 1 = 0$ and perpendicular to the yz plane.

Unit-IV:

Representation of line–angle between a line and a plane – co – planar lines–shortest distance between two skew lines –length of the perpendicular.

4.1 Representation of Line:

4.1.1. A straight line may be determined as the intersection of two planes.

Let the equation of two planes be

$$Ax + By + Cz + D = 0\ , A_1x + B_1y + C_1z + D_1 = 0.$$

Any set of values of x, y, z which satisfy the two equations simultaneously, will give the coordinates of a point, which lies in the line of intersection of the two planes. Hence the equation of planes taken together.

$Ax + By + Cz + D = 0\ , A_1x + B_1y + C_1z + D_1$ give the equation of the line of intersection of the two planes.

Cor. The intersection of XZ and XY planes is the x −axis. Hence the equation of the x −axis is $y = 0, z = 0$.

Similarly, the equation of the $y - axis$ are $x = 0, z = 0$ and of the $z - axis$ are $x = 0, y = 0$.

4.1.2. Symmetrical form of the equations of a line.

Let the direction cosines of a line passing through a given point $A(x_1, y_1, z_1)$ be l, m, n. Let the coordinates of any point p on it be (x, y, z) and let the distance AP be r.

Projecting AP on the coordinate's axes, we get

$$x - x_1 = lr; y - y_1 = mr; z - z_1 = nr.$$

Hence the equations of the line are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r.$$

Cor.1:

The coordinates of any point on this line can be expressed in the term of a single parameter r.

$$x=x_1+lr, y=y_1+mr, z=z_1+nr.$$

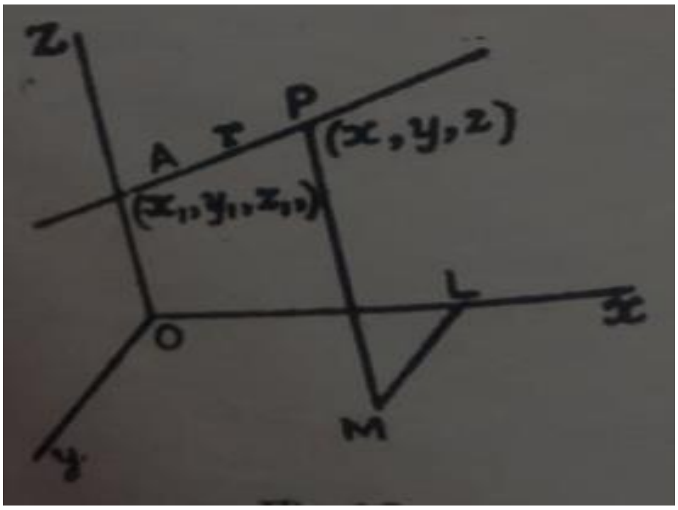
Even if l, m, n are not the direction cosines of the line but only the direction ratios, then also (x,y,z) and (x_1,y_1,z_1) .

Cor.2:

Any equation of the form

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

represents a straight line passing through the point (x_1,y_1,z_1) and whose direction ratios are l,m,n.



Cor.3:

The equations of the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

are only special form of the previous form.

The equation

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}\ and\ \frac{y-y_1}{m}=\frac{z-z_1}{n},$$

represent a pair of planes passing through the line.

4.1.3. The symmetrical form of the equation of the line.

$$ax+by+cz+d=0=\ a_1x+b_1y+c_1z+d_1.$$

To put the equation of the line into the symmetrical form we have to find

(1) the direction ratios of the line; and

(2) the coordinates of any point on it.

Let the direction ratios of the le be l,m,n.

The line is perpendicular to the normal of both the planes, since the line lies in both the planes.

The direction ratios of the normal of the planes respectively a, b, c and a_1,b_1,c_1.

$$\therefore al+bm+cn=0$$

$$a_1l+b_1m+c_1n=0$$

$$\therefore\ \frac{l}{bc_1-b_1c}=\frac{m}{ca_1-c_1a}=\frac{n}{ab_1-a_1b}.$$

Let as find the point where this line meets the XY plane i.e., $z = 0$

The coordinates of that point are given by the equation

$ax + by + d = 0$

$a_1x + b_1y + d_1 = 0$ and $z = 0$.

$$\therefore \frac{x}{bd_1 - b_1d} = \frac{y}{da_1 - d_1a} = \frac{1}{ab_1 - a_1b}.$$

$\therefore$ The coordinates of the point where the line meets the XY plane are

$$\left(\frac{bd_1 - b_1d}{ab_1 - a_1b}, \quad \frac{da_1 - d_1a}{ab_1 - a_1b}, 0\right).$$

$\therefore$ The equations of the line are

$$\frac{x - \dfrac{bd_1 - b_1d}{ab_1 - a_1b}}{bc_1 - b_1c} = \frac{y - \dfrac{bd_1 - b_1d}{ab_1 - a_1b}}{ca_1 - c_1a} = \frac{z}{ab_1 - a_1b}$$

Example.1:

Find the symmetrical form of the equation of the line of intersection of the planes.

$x + 5y - z - 7 = 0, 2x - 5y + 3z + 1 = 0.$

Solution:

The normal to the two planes are in the direction given by

$$1, 5, -1 \; and \; 2, -5, 3.$$

Each of these two directions is perpendicular to that of the line of intersection. I$l:m:n$ are the direction ratios of the line of intersection, we get

$$l + 5m - n = 0$$

$$2l - 5m + 3n = 0$$

$$\therefore \frac{l}{10} = \frac{m}{-5} = \frac{n}{-15}$$

$$i.e., \frac{l}{2} = \frac{m}{-1} = \frac{n}{-3}$$

We have to find coordinates of any fixed point on it and there is an unlimited number of points from which to choose. We shall take the point in each the lie meets the plane $z = 0$.

The $x\ and$ y coordinates of this point are given by

$$x + 5y - 7 = 0$$

$$2x - 5y + 1 = 0$$

$$\therefore x = 2, y = 1.$$

Thus,

one point on this line is $(2, 1, 0)$.

Hence,

the equations of the line are

$$\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z}{-3}$$

4.1.4. Equation of a straight line passing through two given points.

If the given points are (x_1, y_1, z_1) and (x_2, y_2, z_2), the direction ratios of the line passing through them are $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$. Then the equations of the line are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Example 1:

Find the point where the line $\frac{x-2}{2} = \frac{y-4}{-3} = \frac{z+6}{4}$ meets the plane $2x + 4y - z - 20$.

Solution:

Let $\frac{x-2}{2} = \frac{y-4}{-3} = \frac{z+6}{4} = r$

$\therefore$ The coordinates of any point on the line are $(2 + 2r, 4 - 3r, -6 + 4r)$.

If the point lies on the plane $2x + 4y - z - 2 = 0$, we get

$$2(2 + 2r) + 4(4 - 3r) — 6 + 4r - 2 = 0$$

$$i.e., r = 2$$

Hence the coordinates of the required point are $(6, -2, 2)$.

Example 2:

Find the perpendicular distance from $p(3, 9, -1)$ to the line $\frac{x+8}{-8} = \frac{y-31}{1} = \frac{z-13}{5}$.

Solution:

Let the foot of the perpendicular form P to the line be A.

Since A is on the line, its coordinates are $(-8r - 8, r + 31, 5r + 13)$

The direction ratios f the line AP are proportional to

$$8r-8-3, r+31-9, 5r+13+1$$

$$i.e., -8r-11, r+22, 5r+14$$

AP is perpendicular to the given line.

$$\therefore\ -8(-8r-11)+1(r+22)+5(5r+14)=0$$

Simplifying, we get $r=-2$

$\therefore$ A is the point $(8,29,3)$.

$\therefore AP^2 = (8-5)^2+(29-9)^2+(3+1)^2=441$

$\therefore$ AP$=21$.

$\therefore$ The perpendicular distance from A to the line is 21 units.

Example 3:

Find the image of the point $(1,-2,3)$ in the plane $2x-3y+2z+3=0$.

Solution:

Let Q be the image of the point $P(1,-2,3)$ on the plane $2x-3y+2z+3=0$.

The PQ is perpendicular to the plane.

The direction ratio of PQ is proportional to $2,-3,2$.

The equation of the line PQ are

$$\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z-3}{2}$$

The coordinates of Q are of the form

$$\frac{x-1}{2}=\frac{y+2}{-3}=\frac{z-3}{2}=r$$

$x=2r+1 \qquad y=-3r-2 \qquad z=2r+3$

$\therefore$ The coordinates of Q are of the form $(2r+1, -3r-2, 2r+3)$ and $P\ (1, -2, 3)$

The Mid-point of PQ is

$$\left(\frac{1+2r+1}{2}, \quad \frac{-2-3r-2}{2}, \frac{3+2r+3}{2}\right)$$

$$i.e, \ \left(r+1, \frac{-3r-4}{2}, r+3\right)$$

This point lies on the plane $2x - 3y + 2z + 3 = 0$.

$$\therefore \ 2(2r+1) - 3(-3r-2) + 2(2r+3) + 3 = 0$$

$$4r + 11 - \frac{(-9r-12)}{2} = 0$$

$$8r + 22 + 9r + 12 = 0$$

$$17r + 34 = 0$$

$$r = -2.$$

Substitute $r = -2$,

$$\Rightarrow \quad \big((2r+1), -3(-2) - 2, \ \ 2(-2) + 3\big)$$

$$(-3, 4, -1)$$

Hence,

Q is the point $(-3, 4, -1)$

Example 4:

Find the equation of the image of the line $\frac{x-1}{2} = \frac{y+2}{-5} = \frac{z-3}{2}$ in the plane

$2x - 3y + 2z + 3 = 0.$

Solution:

The image of this line in the plane is the straight line joining the image in the plane of two points on the given line.

$(1, -2, -3)$ is the point on the line and its image in the plane $2x - 3y + 2z + 3 = 0$ is $(-3, 4, -1)$ from the above example.

The coordinates of the point R in which the line meets the plane are given by $(1 + 2r, -2 - 5r, 3 + 2r)$,

where,

$$2(1 + 2r) - 3(-2 - 5r) + 2(3 + 2r) + 3 = 0$$

$$23r + 17 = 0$$

$$r = \frac{-17}{23}$$

Substitute $r = \frac{-17}{23}$

$$\Rightarrow \quad (1 + 2r, -2 - 5r, 3 + 2r)$$

$$\left(1 + 2\left(\frac{-17}{23}\right), -2 - 5\left(\frac{-17}{23}\right), 3 + 2\left(\frac{-17}{23}\right)\right)$$

$$\left(\frac{-11}{23}, \frac{39}{23}, \frac{35}{23}\right)$$

$\therefore$ R is the point

$$\left(\frac{-11}{23}, \frac{39}{23}, \frac{35}{23}\right)$$

Hence the reference line is the line joining the points

$(-3, 4, -1)$ & $\left(\frac{-11}{23}, \frac{39}{23}, \frac{35}{23}\right)$

Hence the equations are,

$$\frac{x+3}{\frac{-11}{23}+3} = \frac{y-4}{\frac{39}{23}-4} = \frac{z+1}{\frac{35}{23}+1}$$

$$\frac{23(x+3)}{58} = \frac{-23(y-4)}{53} = \frac{23(z+1)}{58}$$

$$(\div 23) \Rightarrow \frac{x+3}{58} = \frac{-(y-4)}{53} = \frac{z+1}{58}$$

$$i.e., \frac{x+3}{-58} = \frac{y-4}{53} = \frac{z+1}{-58}$$

Example 5:

The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B, C. Find the coordinates of the orthocentre of the ⊿ABC.

Solution:

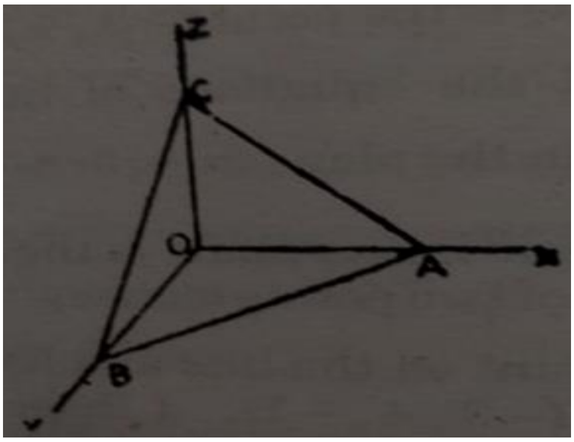
The point A, B, C are respectively $(a, 0,0), (0, b, 0), (0,0, c)$.

The direction cosines of the lie BC are proportional to 0, b, -c.

Hence the equation of the line BC is

$$\frac{x}{a} + \frac{y-b}{b} + \frac{z}{-c}$$

Let the line through A, perpendicular to BC have direction cosines proportional to l, m, n.



Then, its equation is

$$\frac{x-a}{l} = \frac{y}{m} = \frac{z}{n}$$

$$\therefore l(0) + m(b) + n(-c) = 0$$

$$mb - nc = 0$$

$$mb = nc$$

Hence the equation of the line becomes

$$\frac{x-a}{l} = \frac{by}{mb} = \frac{cz}{nc}.$$

Hence the equation of the plane passing through OX perpendicular to BC is by= cz.

Similarly, the equation of the plane through OY perpendicular to CA is, CZ= ax.

These two planes will intersect on the line $ax = by = cz$.

$$i.e., \frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c}$$

Hence the orthocentre is the intersection of this line with the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

We can easily show that the coordinates of the orthocentre are,

$$\frac{1/a}{1/a^2+1/b^2+1/c^2},\frac{1/b}{1/a^2+1/b^2+1/c^2},\frac{1/c}{1/a^2+1/b^2+1/c^2}.$$

4.1.5. The condition for the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ to be parallel to the plane $ax+by+cz+d=0$.

Let the equation of the line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ $\rightarrow$ (1)

& The equation of the plane be $ax+by+cz+d=0$ $\rightarrow$ (2)

If the line (1) is parallel to the plane (2), the line (1) is perpendicular to the normal of the plane (2) $\rightarrow$ (3)

Now, direction ratios of the line (1) = l, m, n.

Direction ratios of the normal to the plane (2) = a, b, c.

$$al+bm+cn=0 \qquad [by(3)]$$

Hence the condition for the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

to be parallel to the plane $ax+by+cz+d=0$ is $al+bm+cn=0$.

$$\therefore r=\frac{-(ax_1+by_1+cz_1+d)}{al+bm+cn}.$$

Cor 1:

The condition for the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ lies in the plane $ax+by+cz+d=0$.

Let the equation of the line be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \qquad \rightarrow (1)$$

& The equation of the plane be $ax+by+cz+d=0$ $\rightarrow (2)$

The line equation (1) will be in the plane equation (2) if

i) the line is parallel to the plane

ii) any one point of the line lies in the plane

(i) By the previous theorem, the line is parallel to the plane (2) if $al+bm+cn=0$

(ii) (1) $\Rightarrow$ $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

$\Rightarrow (x_1,y_1,z_1)$ lie in the plane (1).

Also, this point (x_1,y_1,z_1) lie in the plane (2)

Equation $(2)\Rightarrow ax_1+by_1+cz_1+d=0$

Hence the condition for the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ lies in the plane $ax+by+cz+d=0$ is

i) $al+bm+cn=0$

ii) $ax_1+by_1+cz_1+d=0$.

Cor 2:

The equation of any plane containing the $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ subject the condition $al+bm+cn=0$.

4.2 Angle between a line and a plane:

4.2.1 Angle between the plane $ax + by + cz + d = 0$ *and the line* $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$.

The angle between a line and a plane is defined to be the angle between the line and its projection on the plane.

Let the equation of the line be $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \rightarrow (1)$ & the equation of the plane be $ax + by + cz + d = 0 \quad \rightarrow (2)$

Let θ be the angle between the line (1) & the plane (2).

$\Rightarrow 90° - \theta$ is the angle between the line (1) & the normal to the plane (2)

Now, direction ratios of the line (1) = l, m, n.

Direction ratios of the normal to the plane (2) =a, b, c.

W.K.T,

$$cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\sum a_1^2}.\sqrt{\sum a_2^2}}$$

$$\cos(90° - \theta) = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}.\sqrt{l^2 + m^2 + n^2}}$$

$$\sin\theta = \frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}.\sqrt{l^2 + m^2 + n^2}}$$

$$\theta = \sin^{-1}\left(\frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}.\sqrt{l^2 + m^2 + n^2}}\right).$$

Cor 1:

If the line is parallel to the plane, then $\theta = 0$.

$\therefore \sin\theta = 0$

$$\frac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}.\sqrt{l^2 + m^2 + n^2}} = 0 \Rightarrow al + bm + cn = 0$$

Hence the line is parallel to the plane if, $al + bm + cn = 0$.

Cor 2:

If the line is perpendicular to the plane, then the line is parallel to the normal of the plane.

$\therefore$ Direction ratios of the lines proportional to the direction ratio of the normal to the plan,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

Definition:

The line of the greatest slope in a given plane is a line which lies in the lane and which is perpendicular to the line of intersection of the plane with the horizontal plane.

Example 1:

Find the equation of the orthogonal projection of the line $\frac{x-2}{4} = \frac{y-1}{2} = \frac{z-4}{3}$ on the plane $8x + 2y + 9z - 1 = 0$.

Solution:

The refereed orthogonal projection lies in the plane drawn through the given line perpendicular to the given plane.

The equation of any plane containing the give line is,

$$A(x-2)+B(y-1)+C(z-4)=0 \qquad \rightarrow (1)$$

Sub to the condition, $4A+2B+3C=0 \qquad \rightarrow (2)$

Plane (1) is perpendicular to the plane $8x+2y+9z-1=0$.

$$\therefore 8A+2B+9C=0 \qquad \rightarrow (3)$$

$$\begin{vmatrix} A & B & C \\ 4 & 2 & 3 \\ 8 & 2 & 9 \end{vmatrix}=0 \Rightarrow \frac{A}{18-6}=\frac{-B}{36-24}=\frac{C}{8-16}$$

$$\times 4 \Rightarrow \quad \frac{A}{3}=\frac{B}{-3}=\frac{C}{-2}$$

Sub the value of A, B, C in (1) we get the equation of the plane (1) as,

$$3(x-2)+(-3)(y-1)+(-2)(z-4)=0$$

$$3x-3y-2z+5=0.$$

Example 2:

If l is the line $\frac{x}{-1}=\frac{y-1}{2}=\frac{z+2}{1}$ find the equation of plane through $'l'$ which is parallel to the line of intersection of the planes $5x+2y+3z=4$ & $x-y+5z+6=0$.

Solution:

The equation of any plane passing through l is

$$Ax+B(y-1)+c(x+2)=0, \rightarrow (1)$$

where $-A+2B+C=0 \qquad \rightarrow (2)$

Let l, m, n be the direction ratio of the line of intersection of the planes $5x + 2y + 3z = 4$ & $x - y + 5z + 6 = 0$

Then,

$$5l + 2m + 3n = 0 \quad \rightarrow (3)$$

$$l - m + 5n = 0 \quad \rightarrow (4)$$

$$\begin{vmatrix} l & m & n \\ 5 & 2 & 3 \\ 1 & -1 & 5 \end{vmatrix} = 0 \Rightarrow \frac{l}{10+3} = \frac{m}{25-3} = \frac{n}{-5-2}$$

$$\frac{l}{13} = \frac{m}{-22} = \frac{n}{-7}$$

From (3) & (4) we get,

$$\frac{l}{13} = \frac{m}{-22} = \frac{n}{-7}$$

Hence the plane (1) is parallel to the line whose direction ratios are proportional to $13, -22, -7$.

$$\therefore 13A - 22B - 7C = 0 \quad \rightarrow (5)$$

From (2) & (5)

$$\begin{vmatrix} A & B & C \\ -1 & 2 & 1 \\ 13 & -22 & -7 \end{vmatrix} = 0 \Rightarrow \frac{A}{-14+22} = \frac{B}{7-13} = \frac{C}{22-26}$$

$$\frac{A}{8} = \frac{B}{6} = \frac{C}{-4}$$

Sub the value of A, B, C in (1) we get the required plane as,

$$Ax + B(y-1) + C(z+2) = 0$$

$$8x + 6(y-1) - 4(z+2) = 0$$

$$8x + 6y - 4z - 14 = 0$$

$$\div 2 \Rightarrow 4x + 3y - 2z - 7 = 0.$$

4.3 Coplanar lines:

4.3.1. The condition that the two given straight lines should be coplanar.

Let this equation be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \rightarrow (1)$$

$$\frac{x-x_2}{l_1} = \frac{y-y_2}{m_1} = \frac{z-z_2}{n_1} \quad \rightarrow (2)$$

The equation to a plane through the first line

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0, \quad \rightarrow (3)$$

where $Al + Bm + Cn = 0. \quad \rightarrow (4)$

If it contains the line (2), then the point(x_2, y_2, z_2) lies on it.

$$\therefore A(x_2-x_1) + B(y_2-y_1) + C(z_2-z_1) = 0 \quad \rightarrow (5)$$

Also, the line (2) is perpendicular to normal to the plane (3)

$$\therefore Al_1 + Bm_1 + Cn_1 = 0 \quad \rightarrow (6)$$

Eliminating A, B, C from equation (5), (4), (6) we get the condition for the lines to be coplanar.

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Eliminating A, B, C from (3), (4), (6) we get the equation of the plane passing through two lines as,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Another Method:

Let the two given lines be,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r\ (say) \quad \rightarrow (1)$$

$$\frac{x - x_2}{l_1} = \frac{y - y_2}{m_1} = \frac{z - z_2}{n_1} = r_1\ (say) \quad \rightarrow (2)$$

W.K.T,

two coplanar lines must be either parallel (or) intersecting, the lines (1) & (2) are parallel if.

$$\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1} \quad \rightarrow (3)$$

Suppose the lines (1) & (2) are not parallel, then they will still be coplanar, if they interest.

The coordinates of any point on the line (1) are

$$(x_1 + lr, y_1 + mr, z_1 + nr) \quad \rightarrow (4)$$

The coordinates of any point on the line (2) are

$$(x_2 + l_1 r_1, y_2 + m_1 r_1, z_2 + n_1 r_1) \quad \rightarrow (5)$$

The lines (1) & (2) intersect, if some particular values of r & r_1 the point (4) & (5) must coincide.

$$\therefore \; x_1 + lr = x_2 + l_1 r_1 \;\Rightarrow\; (x_1 - x_2) + lr - l_1 r_1 = 0 \;\rightarrow (6)$$

$$y_1 + mr = y_2 + m_1 r_1 \;\Rightarrow\; (y_1 - y_2) + mr - m_1 r_1 = 0 \;\rightarrow (7)$$

$$z_1 + nr = z_2 + n_1 r_1 \;\Rightarrow\; (z_1 - z_2) + nr - n_1 r_1 = 0 \;\rightarrow (8)$$

Eliminating r & r_1 from equation (6), (7) & (8).

$$\begin{vmatrix} x_1 - x_2 & l & l_1 \\ y_1 - y_2 & m & m_1 \\ z_1 - z_2 & n & n_1 \end{vmatrix} = 0$$

$$i.e., \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0.$$

Example 1:

Find the condition for the lines $ax + by + cz + d = 0 = a_1x + b_1y + c_1z + d_1$ and $a_2x + b_2y + c_2z + d_2 = 0 = a_3x + b_3y + c_3z + d_3$ to be the coplanar.

Solution:

Let the lines intersect at the point (x_1, y_1, z_1)

Then (x_1, y_1, z_1) lies on the planes

$$ax+by+cz+d=0$$

$$a_1x+b_1y+c_1z+d_1=0$$

$$a_2x+b_2y+c_2z+d_2=0$$

$$a_3x+b_3y+c_3z+d_3=0$$

$$\therefore\ a_1x+b_1y+c_1z+d_1=0$$

$$a_1x_1+b_1y+c_1z+d_1=0$$

$$a_2x_1+b_2y+c_2z+d_2=0$$

$$a_3x_1+b_3y+c_3z+d_3=0$$

Eliminating x_1, y_1, z_1 from the above four equation, we get the condition

$$\begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$$

Example 2:

Prove that the lines

$$\frac{x+1}{-3}=\frac{y+10}{8}=\frac{z-1}{2}\ ;\ \frac{x+3}{-4}=\frac{y+1}{7}=\frac{z-4}{1}$$

are coplanar. Find also their point of intersection and the plane through them.

Solution:

Let $\frac{x+1}{-3}=\frac{y+10}{8}=\frac{z-1}{2}=r \quad \rightarrow (1)$

$$\frac{x+3}{-4}=\frac{y+1}{7}=\frac{z-4}{1}=r_1 \quad \rightarrow (2)$$

The co-ordinates of the points on the two lines are respectively of the form

$$(-3r-1, 8r-10, 2r+1) \quad \rightarrow (3)$$

and $(-4r_1-3, 7r_1-1, r_1+4) \quad \rightarrow (4)$

The lines are coplanar if the lines intersect.

i.e., if the three equations

$$-3r-1 = -4r_1-3$$

$$8r-10 = 7r_1-1$$

$$2r+1 = r_1+4$$

Solving the first two equations, we get $r = 2$ *and* $r_1 = 1$.

These values satisfy the third equation also.

∴ The lines are coplanar.

Substituting the value of r in (1) or in the value of r_1 in (2), we get the co-ordinates of the intersecting point.

The intersecting point is $(-7, 6, 5)$.

The equation of the plane containing the lines is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+1 & y+10 & z-1 \\ -3 & 8 & 2 \\ -4 & 7 & 1 \end{vmatrix} = 0$$

$$i.e., 6x+5y-11z+67 = 0$$

4.4 The shortest distance between two skew lines:

4.4.1. The shortest distance between two given lines.

Let the given lies AB and $A'B'$ whose equations are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}\ and\ \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

Let the shortest distance between the lines have direction cosines, l,m,n.

The shortest distance GH is perpendicular to both the lines.

$$\therefore ll_1+mm_1+nn_1=0$$

$$ll_2+mm_2+nn_2=0$$

$$\therefore \frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-l_1n_2}=\frac{n}{l_1m_2-l_2m_1}$$

$$=\frac{1}{\sqrt{\{\sum(m_1n_2-m_2n_1)^2}}$$

Let the point A be

$$(x_1,y_1,z_1),A'(x_2,y_2,z_2)$$

GH=projection of AA' on GH= $(x_2-x_1)l+\ (y_2-y_1)m+\ (z_2-z_1)n$

$$=\frac{(x_2-x_1)m_1n_2-m_2n_1+(y_2-y_1)n_1l_2-l_1n_2+(z_2-z_1)l_1m_2-l_2m_1}{\sqrt{\{\sum(m_1n_2-m_2n_1)^2}}$$

$$=\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}\div\sqrt{\{\sum(m_1n_2-m_2n_1)^2}$$

The shortest distance is the line of intersection of the planes containing the lines AB and GH; and $A'B'$ and GH.

Hence GH is the line

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

Cor:

The two lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \ and \ \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

are coplanar if the shortest distance between them is zero.

$$i.e., \quad \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

4.5 Length of the perpendicular:

Example 1:

Find the shortest distance between the lines

$$\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}; \ \frac{x-1}{1} = \frac{y+1}{3} = \frac{z+2}{2}$$

Solution:

Let the directions cosines of the line perpendicular to both the lines be l, m, n.

Then $-l + 2m + n = 0$

$l + 3m + 2n = 0$

$$\therefore \quad \frac{l}{1} = \frac{m}{3} = \frac{n}{-5}.$$

$$\therefore \; l = \frac{1}{\sqrt{35}}\,, m = \frac{3}{\sqrt{35}}, n = \frac{-5}{\sqrt{35}}$$

The magnitude of the shortest distance is the projection of the line joining the points $(3,4,-2)$ and $(1,-7,-2)$ on the line of shortest distance.

$$\therefore S.D = (3-1)\frac{1}{\sqrt{35}} + (4+7)\frac{3}{\sqrt{35}} + (-2+2)\frac{1}{\sqrt{35}}$$

$$= \sqrt{35}.$$

The equation of the shortest distance between them is

$$\begin{vmatrix} x-3 & y-4 & z+2 \\ -1 & 2 & 1 \\ 1 & 3 & -5 \end{vmatrix} = 0 = \begin{vmatrix} x-1 & y+7 & z+2 \\ 1 & 3 & 2 \\ 1 & 3 & -5 \end{vmatrix}$$

Simplifying, we get $13x + 4y + 5z - 45 = 0 = 3x - y - 10.$

Another method:

$P(3-r, 4+2r, -2+r)$ and $Q(1+r_1, -7+3r_1, -2+2r_1)$ are the general co-ordinates of the two points on the two lines respectively. The direction cosines of PQ are proportional to

$$2 - r - r_1, 11 + 2r - 3r_1, r - 2r_1.$$

PQ is the shortest distance between the two lines. Hence it is perpendicular to both the lines.

$$\therefore\ (2-r-r_1)(-1)+(11+2r-3r_1)(2)+(r-2r_1)=0$$

$$(2-r-r_1)(1)+(11+2r-3r_1)3+(r-2r_1)2=0$$

$$i.e., 6r-7r_1+20=0$$

$$7r-14r_1+35=0$$

$$\therefore r=-1, r_1=2.$$

Hence the co-ordinates of P and Q are $(4,2,3)$ *and* $(2,-1,2)$.

$\therefore$ The distance $PQ = \sqrt{(4-3)^2+(2+1)^2+(-3-2)^2}$

$$=\sqrt{35}.$$

The equation of the line PQ is

$$\frac{x-4}{1}=\frac{y-2}{3}=\frac{z+3}{-5}$$

Example 2:

Prove that the shortest distance between the lines

$$ax+by+cz+d=0=a_1x+b_1y+c_1z+d_1 \quad \rightarrow (1)$$

$$\alpha x+\beta y+\gamma c+\delta=0=\alpha_1 x+\beta_1 y+\gamma_1 c+\delta_1 \quad \rightarrow (2)$$

is

$$\begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ \alpha & \beta & \gamma & \delta \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \end{vmatrix} \div \{\sum(BC'-B'C)^2\}^{1/2}$$

where $A=bc_1-b_1c, A'=\beta r_1=-\beta_1 r, etc.,$

the equation of any plane passing through the line (1) is

$$ax + by + cz + d + k(a_1x + b_1y + c_1z + d_1) = 0$$

$$i.e., (a + ka_1)x + (b + kb_1)y + (c + kc_1)z + (d + kd_1) = 0. \dots.(3)$$

The equation of any plane passing through the line (2) is

$$\alpha x + \beta y + \gamma c + \delta + k_1(\alpha_1 x + \beta_1 y + \gamma_1 c + \delta_1)$$

$$i.e., (\alpha + k_1\alpha_1)x + (\beta + k_1\beta_1)y + (\gamma + k_1\gamma_1)z + (\delta + k_1\delta_1) = 0. \quad \dots.(4)$$

The shortest distance D between lines (1) and (2) is the difference between perpendicular to the planes (3) and (4) if the planes are parallel. Planes (3) and (4) are parallel if

$$\frac{a + ka_1}{\alpha + k_1\alpha_1} = \frac{b + kb_1}{\beta + k_1\beta_1} = \frac{c + kc_1}{\gamma + k_1\gamma_1} = \lambda(say)$$

$$a + ka_1 - \lambda\alpha - \lambda k_1\alpha_1 = 0 \quad \dots.(5)$$

$$b + kb_1 - \lambda\beta - \lambda k_1\beta_1 = 0 \quad \dots\dots.(6)$$

$$c + kc_1 - \lambda\gamma - \lambda k_1\gamma_1 = 0 \quad \dots\dots(7)$$

Also,

$$D = \frac{d + kd_1}{\sqrt{\{(a + ka_1)^2 + (b + kb_1)^2 + (c + kc_1)^2\}}} - \frac{\delta + k_1\delta_1}{\sqrt{\{(\alpha + k_1\alpha_1)^2 + (\beta + k_1\beta_1)^2 + (\gamma + k_1\gamma_1)^2\}}}$$

$$= \frac{d + kd_1}{\lambda\sqrt{\{(a + ka_1)^2 + (b + kb_1)^2 + (c + kc_1)^2\}}} - \frac{\delta_1 + k_1\delta_1}{\sqrt{\{(\alpha + k_1\alpha_1)^2 + (\beta + k_1\beta_1)^2 + (\gamma + k_1\gamma_1)^2\}}}$$

The direction cosines of the line (1) are proportional to

$$(bc_1 - b_1c), \qquad (ca_1 - c_1a), \qquad (ab_1 - a_1b)$$

$$i.e., \; A \quad , \quad B \, , \quad C.$$

This line is parallel to the plane (4).

$$\therefore A(\alpha + k_1\alpha_1) + B(\beta + k_1\beta_1) + C(\gamma + k_1\gamma_1) = 0$$

$$\therefore \; k_1 = -\frac{A\alpha + B\beta + c\gamma}{(A\alpha_1 + B\beta_1 + c\gamma_1)}$$

$$\therefore \; \alpha + k_1\alpha_1 = \; \alpha - \frac{\alpha_1(A\alpha + B\beta + c\gamma)}{(A\alpha_1 + B\beta_1 + c\gamma_1)}$$

$$= (BC' - B'C) \div (A\alpha_1 + B\beta_1 + c\gamma_1)$$

Similarly, $\beta + k_1\beta_1 = CA' - C'A \; \div \; (A\alpha_1 + B\beta_1 + c\gamma_1)$

$$\gamma + k_1\gamma_1 = AB' - A'B \; \div \; (A\alpha_1 + B\beta_1 + c\gamma_1)$$

$$\therefore \; (\alpha + k_1\alpha_1)^2 + (\beta + k_1\beta_1)^2 + (\gamma + k_1\gamma_1)^2 = \frac{\sum(BC' - B'C)^2}{(A\alpha_1 + B\beta_1 + c\gamma_1)^2}$$

$$= \mu^2 \text{(say)}$$

$$\therefore D = \frac{d + kd_1}{\lambda\mu} - \frac{\delta + k_1\delta_1}{\mu}$$

$$\therefore \; d + kd_1 - \lambda(\delta - D\mu) - \lambda k_1\delta_1 = 0 \qquad \ldots\ldots (8)$$

Eliminating $k, -\lambda, -\lambda\kappa_1$ from (5), (6), (7) and (8) we get

$$\begin{vmatrix} a & a_1 & \alpha & \alpha_1 \\ b & b_1 & \beta & \beta_1 \\ c & c_1 & \gamma & \gamma_1 \\ d & d_1 & (\delta - D\mu) & \delta_1 \end{vmatrix} = 0$$

$$i.e., \begin{vmatrix} a & a_1 & \alpha & \alpha_1 \\ b & b_1 & \beta & \beta_1 \\ c & c_1 & \gamma & \gamma_1 \\ d & d_1 & \delta & \delta_1 \end{vmatrix} - D\mu \begin{vmatrix} a & a_1 & \alpha_1 \\ b & b_1 & \beta_1 \\ c & c_1 & \gamma_1 \end{vmatrix} = 0$$

$$\therefore D = \begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ \alpha & \beta & \gamma & \delta \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \end{vmatrix} \div \mu \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

$$But \begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} = \alpha_1(bc_1 - b_1c) + \beta_1(ca_1 - c_1a) + \gamma_1(ab_1 - a_1b)$$

$$= (A\alpha_1 + B\beta_1 + c\gamma_1)$$

$$\therefore D = \begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ \alpha & \beta & \gamma & \delta \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \end{vmatrix} \div \mu(A\alpha_1 + B\beta_1 + c\gamma_1)$$

$$= \begin{vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ \alpha & \beta & \gamma & \delta \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \end{vmatrix} \div \{\sum(BC' - B'C)^2\}^{1/2}$$

Example 3:

The straight lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}; \frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$$

are cut by a third whose direction cosines are λ, μ, ν. Show that the length intercepted on the third line is given by $\begin{vmatrix} \alpha-\alpha_1 & \beta-\beta_1 & \gamma-\gamma_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \lambda & \mu & \nu \end{vmatrix}$

and deduce the length of the shortest distance.

Solution:

The general co-ordinates of points on the two lines are respectively

$$(\alpha + lr, \beta + mr, \gamma + nr) \; and (\alpha_1 + l_1r_1, \beta_1 + m_1r_1, \gamma_1 + n_1r_1)$$

If d is the intercept on the third line and if P, Q are the points of intersection of the lines on the third line, projection of PQ on the x-axis is

$$d\lambda = (\alpha + lr) - (\alpha_1 + l_1r_1)$$

Similarly, $d\mu = (\beta + mr) - (\beta_1 + m_1r_1)$

$$d\nu = (\gamma + nr) - (\gamma_1 + n_1r_1)$$

$$i.e., (\alpha - \alpha_1) + lr - l_1r_1 - d\lambda = 0$$

$$(\beta - \beta_1) + mr - m_1r_1 - d\mu = 0$$

$$(\gamma - \gamma_1) + nr - n_1r_1 - d\nu = 0$$

Hence solving these equations considering them as equation containing r, r_1 and d_1, we get,

$$\frac{r}{\begin{vmatrix} l_1 & \lambda & \alpha-\alpha_1 \\ m_1 & \mu & \beta-\beta_1 \\ n_1 & \nu & \gamma-\gamma_1 \end{vmatrix}} = \frac{r_1}{\begin{vmatrix} \lambda & \alpha-\alpha_1 & l \\ \mu & \beta-\beta_1 & m \\ \nu & \gamma-\gamma_1 & n \end{vmatrix}}$$

$$= \frac{d}{\begin{vmatrix} \alpha-\alpha_1 & l & l_1 \\ \beta-\beta_1 & m & m_1 \\ \gamma-\gamma_1 & n & n_1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} l & l_1 & \lambda \\ m & m_1 & \mu \\ n & n_1 & \nu \end{vmatrix}}$$

$$\therefore d = \begin{vmatrix} \alpha-\alpha_1 & l & l_1 \\ \beta-\beta_1 & m & m_1 \\ \gamma-\gamma_1 & n & n_1 \end{vmatrix} \div \begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \lambda & \mu & \nu \end{vmatrix}$$

This intercept will be the shortest distance if the third line is perpendicular to both the lines.

$$\therefore l\lambda + m\mu + n\nu = 0$$

$$l_1\lambda + m_1\mu + n_1\nu = 0$$

$$\therefore \frac{\lambda}{mn_1 - m_1n} = \frac{\mu}{nl_1 - l_1n} = \frac{\nu}{lm_1 - l_1m} = \frac{1}{\{\sum(mn_1 - m_1n)^2\}^{1/2}}$$

$$\therefore \begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \lambda & \mu & \nu \end{vmatrix} = \lambda(mn_1 - m_1n) + \mu(nl_1 - l_1n) + \nu(lm_1 - l_1m)$$

$$= \frac{(mn_1 - m_1n)^2 + (nl_1 - l_1n)^2 + (lm_1 - l_1m)^2}{\{\sum(mn_1 - m_1n)^2\}^{1/2}}$$

$$= \sqrt{\sum(mn_1 - m_1n)^2}$$

$$\therefore S.D. = \begin{vmatrix} \alpha-\alpha_1 & \beta-\beta_1 & \gamma-\gamma_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} \div \sqrt{\sum(mn_1 - m_1n)^2}$$

4.5.1. If $u_1 = 0 = v_1$ *and* $u_2 = 0 = v_2$ be two straight lines, then the general equations of a straight line intersecting them both are $u_1 + \lambda_1 v_1 = 0 = u_1 + \lambda_1 v_1$, where λ_1, λ_2 are constants.

The line $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$ lies in the plane $u_1 + \lambda_1 v_1 = 0$ which again contains the line $u_1 = 0 = v_1$.

The two lines $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$ and $u_1 = 0 = v_1$ are therefore coplanar and hence they intersect.

Similarly, the same line intersects the line $u_2 = 0 = v_2$.

4.5.2. The equations of two skews lines in a simplified form.

Let the shortest distance between two given lines AB and CD meet them at E and F and be of length 2c. Bisect EF at O and draw $A'OB', C'OD'$, parallel

AB, CD respectively. Then take as axes of x and y the interior and exterior bisectors of the angle $A'OD'$, the axis of z being the line EF. These three lines are mutually at right angles.

If the angle between the given lines be 2α, the line OD' makes angles $\alpha, \frac{\pi}{2}, -\alpha, \frac{\pi}{2}$

With the axes OX, OY, OZ so that the direction cosines of CD which is parallel to OD' are $\cos\alpha, \sin\alpha, 0$. Also since OA' makes angles $\alpha, \frac{\pi}{2} + \alpha, \frac{\pi}{2}$ with the axes, the direction cosines of AB are $\cos\alpha, -\sin\alpha, 0$.

Also, since EF=2c, the co-ordinates of E, F are $(0,0,c)$ and $(0,0,-c)$ respectively.

Thus, the equations to AB, CD are

$$\frac{x}{\cos\alpha} = \frac{y}{-\sin\alpha} = \frac{z-c}{0}, i.e., y = -x\tan\alpha, z = c$$

and

$$\frac{x}{\cos\alpha} = \frac{y}{\sin\alpha} = \frac{z+c}{0}, i.e., y = x\tan\alpha, z = -c.$$

Note 1:

$(r, -rtan\alpha, c) and\ (p, ptan\alpha, -c)$ are the general co-ordinates of points on the two lines, r and p being any two constants.

Note 2:

Solutions to problems relating to two non-intersecting given straight lines are often simplified by taking the equations of the line in the simplified form obtained above.

Note 3:

Putting tan α= m, we can also express the equations of the lines in the forms $y = -mx, z = c\ and\ y = mx, z = -c.$

Hence the general co-ordinates f points on the two lines are respectively $(r, -mr, c)\ and\ (p, mp, -c)$.

Exercises:

1.Find the equation of the straight line joining the points

(i) $(2,5,8)$ and $(-1,6,3)$.

(ii) $(2,3,7)$ and $(2, -5,8)$.

(iii) Origin and $(5, -2, 3)$.

2. Find the equation of the edges of the tetrahedron whose vertices are at the points $(0,0,0), (0,2,0), (3,2,0), (1,1,2)$.

3. Find the equation of the line passing through the point $(3,2,-8)$ and its perpendicular to the plane $3x + y + 2z - 2 = 0$.

4. Find the equation of the plane passing through the point $(3,-2,1)$ and its perpendicular to the line $3x - 5y - 2z + 6 = 0; 4x + y + 3z - 7 = 0$.

5. Find the equation to the line through the point $(2,3,1)$ parallel to the line $-x + 2y + z = 5; x + y + 3z = 6$.

6.Show that $2x + y + 3z - 7 = 0 = x - 2y + z - 5$ and $4x + 4y - 8z = 0 = 10x - 8y + 7z$ are at right angles.

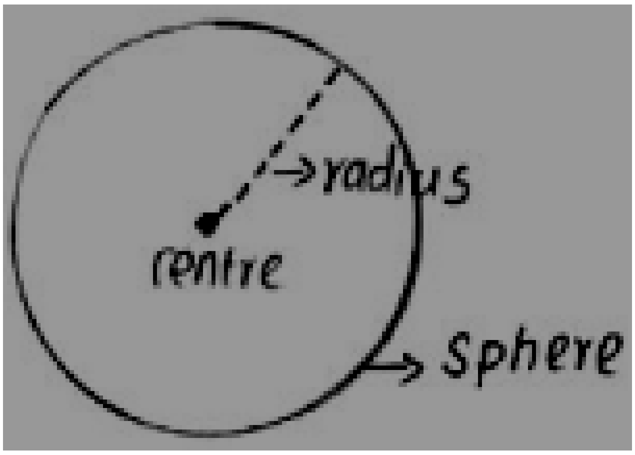
Unit-V:

Equation of a sphere-general equation-section of a sphere by a plane-equation of the circle- tangent plane- angle of intersection of two spheres- condition for the orthogonality.

5.1 Equation of a Sphere:

Definition:

A sphere is the locus of a point which moves such that its distance from a fixed point is always equal to a constant.

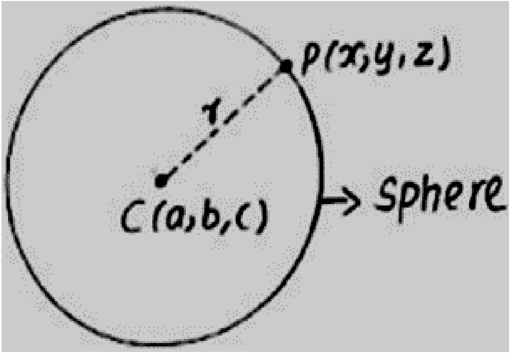


The fixed point is called the "centre" of the sphere and the constant distance is known as "radius" of the sphere.

5.1.1 Centre radius form of a sphere:

The equation of the sphere when the centre and radius are given

Let C (a, b, c) be the centre of the sphere and 'r' be the radius of the sphere.



Let $P\ (x\ ,y\ ,z)$ be any point on the sphere

By definition $CP = r$

$\Rightarrow\ CP2 = r^2$

By the distance formula, $CP^2=(x-a)^2+(y-b)^2+(z-c)^2$

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

This is the required equation of the sphere with centre C (a, b, c) with radius ‘r’.

Corollary:

The equation of the sphere whose centre is the origin and radius ‘a’ is

$x^2 +y^2 +z^2=a^2$.

5.2 General form of equation of the sphere:

The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ always represents a sphere and to

To find its centre and radius:

The given equation is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \_\_\_\_(1)$$

$$\Rightarrow (x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) = -d$$

$$\Rightarrow \ (x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = u^2 + v^2 + w^2 - d$$

$$\Rightarrow (x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$$

$$\Rightarrow (x - (-u))^2 + (y - (-v))^2 + (z - (-w))^2 = (\sqrt{u^2 + v^2 + w^2 - d})2$$

Compare this equation to the sphere equation in general form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

we get,

$$a = -u\,; \quad b = -v; \quad c = -w; \qquad r = \sqrt{u^2 + v^2 + w^2 - d}$$

Hence the equation

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere with centre= $(-u, -v, -w)$ &

radius=$\sqrt{u^2 + v^2 + w^2 - d}$

Note 1:

- If $u^2+v^2+w^2> d$, then $\sqrt{u^2 + v^2 + w^2 - d}$ is real and so the equation (1)

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a real sphere.

- If $u^2+v^2+w^2<d$, then $\sqrt{u^2 + v^2 + w^2 - d}$ is imaginary and so the equation (1)

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a imaginary sphere.

- If $u^2+v^2+w^2=d$, then $\sqrt{u^2 + v^2 + w^2 - d}=0$ and in this case, the sphere (1)

reduces to a point at its centre.

$(i.e.,)$ the sphere (1) reduces to

$$(x+u)2 \; + \; (y+v)2+(z+w)2=0$$

This is called a point sphere and the only real solution of the equation is

$x=-u, y=-v, z=-w.$

Note 2:

5.2.1 Conditions for a sphere:

The equation of a sphere has the following three characteristics:

(i) It is the second degree in x, y, z

(ii) The co-efficient of x^2, y^2, z^2 are all equal.

(iii) The product terms xy , yz , zw are absent.

Note 3:

The most general equation of the second degree in x, y and z is

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ ------------------(*)

This all represent a sphere only if

(i) a=b=c & (ii) f=g=h=0

The equation (*) reduces to

$*\Rightarrow ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$

$\Rightarrow a(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$

$\Rightarrow x^2+y^2+z^2+\frac{2u}{a}x+\frac{2v}{a}y+\frac{2w}{a}z+\frac{d}{a}=0$

The centre of the sphere = $\left(-\frac{u}{a},-\frac{v}{a},-\frac{w}{a}\right)$.

The radius of the sphere= $\sqrt{u^2+v^2+w^2-d}$.

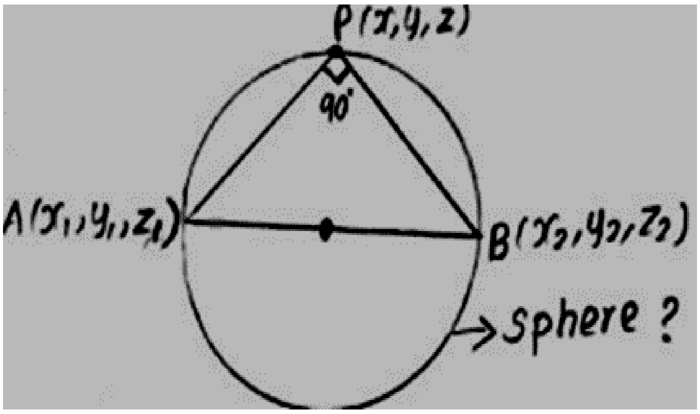
Note 4:

The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is said to be the standard form of the equation of the sphere.

Here the co-efficient of x^2, y^2, z^2 are all equal and each=1.

This equation contains four independent arbitrary constants u, v, w, d.

Diameter form of a sphere



Equation of the sphere described on the line joining the points

A (x_1, y_1, z_1) and B (x_2, y_2, z_2) as diameter.

Let $P(x, y, z)$ be a point on the sphere described on AB as diameter

Directional ratios of AP=$x-x_1$, $y-y_1$, $z-z_1$

Directional ratios of BP=$x-x_2$, $y-y_2$, $z-z_2$

The circle passing through the three points A, B, C and P will have AB for its diameter.

$\therefore \angle P=90°$

(i. e,) AP ⊥to BP

$\therefore a_1a_2+ b_1b_2+ c_1c_2=0$

$\Rightarrow (x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0,$

which is the required equation of the sphere.

Hence the equation of a sphere described on the line joining the points

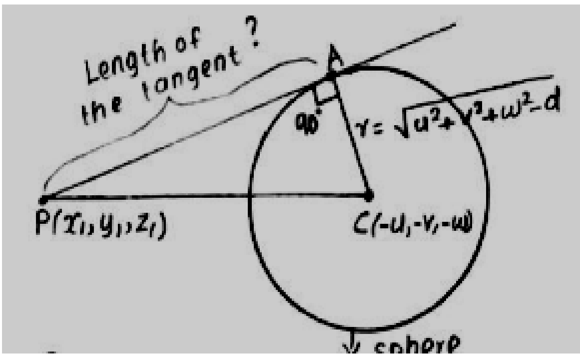
A (x_1, y_1, z_1) and B (x_2, y_2, z_2) as diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

Length of the tangent from an External point to a sphere

5.2.2 The length of the tangent from the point (x_1,y_1,z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Proof:



Let the given sphere be $x^2+y^2+z^2+2ux+2vy+2zw+d=0$ ------(1)

Centre of the sphere, c = (-u, -v, -w)

Radius of the sphere, $r=\sqrt{u^2+v^2+w^2-d}$

(i.e,) $CA=\sqrt{u^2+v^2+w^2-d}$--------------(2)

Let P (x, y, z) be any point outside the sphere (1)

Let PA be the tangent from the point P (x_1, y_1, z_1) to the sphere (1)

ΔCAP is a right-angle triangle (PA perpendicular to CA)

$\Rightarrow PC^2=PA^2+CA^2$

$\Rightarrow (x_1+u)^2+(y_1+v)^2+(z_1+w)^2=PA^2+(u^2+v^2+w^2-d)$ [by distance formula by (2)]

$\Rightarrow x^2 +2ux_1+u^2+y^2+2vy_1+v^2+z^2+2wz_1+w^2=PA^2+u^2+v^2+w^2-d$

$\Rightarrow PA^2=x^2+y^2+z^2+2ux_1+2vy_1+2wz_1+d$

$\Rightarrow PA=(x^2+y^2+z^2+2ux_1+2vy_1+2wz_1+d)^{1/2}$,

which is the required length of the tangent from the point (x_1,y_1,z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

Note:

The value of PA^2 is called the power of P with respect to the circle.

Corollary 1:

The point (x_1, y_1, z_1) lies outside, on (or) inside the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ according as $x^2+y_1+z_1+2ux+2vy+2wz+d >=< 0$.

Corollary 2:

* If 'd' is positive, the origin lies outside the sphere.
* If 'd' is negative, the origin lies inside the sphere.
* If d=0 lies on the sphere.

Example 1:

Find the equation of the sphere with the centre (-1, 2, -3) and radius 3 units.

Solution:

Given centre, (a,b,c)=(-1,2,-3)

Radius r=3 units of the sphere.

The equation of the sphere is

$$(x-a)^2+(y-b)^2+(z-c)^2= r^2$$

$$(x+1)^2+(y-2)^2+(z+3)^2=3^2$$

$$x^2+2x+1+y^2-4y+4+z^2+6z+9 \quad =9$$

$$x^2+y^2+z^2+2x-4y+6z+5 =0$$

Example 2:

Find the co-ordinates of the centre and the radius of the sphere

$2x^2+2y^2+2z^2-2x+4y+2z-15=0$

Solution:

Given the sphere is $2x^2+2y^2+2z^2-2x+4y+2z-15=0$

$$x^2+y^2+z^2-x+2y+z-\frac{15}{2}=0\text{--------------}(1)$$

Compare the sphere (1) to the general form of the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$

we get,

$$2u=-1; \quad 2v=2; \quad 2w=1 \ \& \ d=\frac{-15}{2}$$

$$u=-1/2; \quad v=1; \quad w=1/2 \ \& \ d=\frac{-15}{2}$$

Centre of the sphere (1) = (-u, -v, -w) = $(\frac{-1}{2},-1,\frac{-1}{2})$

Radius of the sphere (1) $=\sqrt{u^{2}+v^{2}+w^{2}-d}$

$$=\sqrt{\frac{1}{4}+1+\frac{15}{2}+\frac{1}{4}}$$

$$=\sqrt{\frac{1+4+1+30}{4}}$$

$$=\sqrt{\frac{36}{4}}$$

$$=\sqrt{9}$$

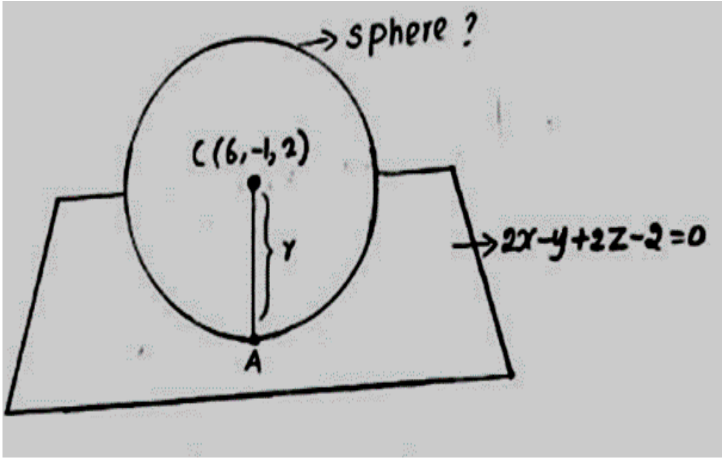
$$= 3 \text{ units}$$

Hence centre = $(\frac{1}{2},-1,\frac{1}{2})$ & radius =3 units.

Example 3:

Find the equation of the sphere which has its centre at the point (6, -1, 2) and touches the plane 2x-y+2z-2=0

Solution:



Let the equation of the sphere be

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$ ----------------(1)

where (a, b, c) is the centre of the sphere

r is the radius of the sphere

Given (a, b, c)=(6,-1,2)

Now,

To find: the radius 'r'

Given the sphere (1) touches the plane 2x-y+2z-2=0

The radius of the sphere (1) is the perpendicular distance from the point C (6, -1, 2) to the plane 2x-y+2z-2=0

W.K.T, the perpendicular distance from the point (x_1,y_1,z_1) to the plane ax+by+cz+d=0 is

$$= \pm\left(\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}}\right)$$

$$\text{Radius } r = \pm\left(\frac{2(6)-(-1)+2(2)-2}{\sqrt{2^2+(-1)^2+2^2}}\right)$$

$$= \pm\left(\frac{12+1+2}{\sqrt{4+1+4}}\right.$$

$$= \pm\left(\frac{15}{3}\right)$$

$$= \pm 5$$

$$\therefore \text{Radius } r = 5$$

The equation of the sphere is

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

$(x-6)^2+(y+1)^2+(z-2)^2=5^2$

$x^2-12x+36+y^2+2y+1+z^2-4z+4=25$

$x^2+y^2+z^2-12x+2y-4z+16=0$

Example 4:

Find the equation of the sphere through the four points (2,3,1), (5,-1,2), (4,3,-1) and (2,5,3).

Solution:

Let the equation of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ -------------- (1)

Given that the sphere (1) passes through the four points

(2,3,1), (5,-1,2) ,(4,3,-1) and (2,5,3)

These four points satisfy the equation of the sphere (1)

$$(1) \Rightarrow \quad 2^2+3^2+1^2+2u(2)+2v(3)+2w(1)+d=0$$

$$5^2+(-1)^2+2^2+2u(5)+2v(-1)+2w(2)+d=0$$

$$4^2+3^2+(-1)^2+2u(4)+2v(3)+2w(-1)+d=0$$

$$2^2+5^2+3^2+2u(2)+2v(5)+2w(3)+d=0$$

$$\Rightarrow \quad 4+9+1+4u+6v+2w+d=0$$

$$25+1+4+10u-2v+4w+d=0$$

$$16+9+1+8u+6v-2w+d=0$$

$$4+25+9+4u+10v+6w+d=0$$

$$\Rightarrow 4u+6v+2w+d+14=0 \text{ --------------(2)}$$

$$10u-2v+4w+d+30=0 \text{ -------------(3)}$$

$$8u+6v-2w+d+26=0 \text{ --------------(4)}$$

$4u+10v+6w+d+38=0$ ------------(5)

(2)$\Rightarrow$ $4u+6v+2w+d+14=0$(7)$\Rightarrow$ $u-w+3=0$

(3) $\Rightarrow$ $10u-2v+4w+d+30=0$(8) $\Rightarrow$ $v+w+6=0$

(-) (+) (-) (-) --------------

---------------------------$u+v+9=0$ -----------(10)

$-6u+8v-2w-16=0$

$3u-4v+w+8=0$--------(6)

(9) $\Rightarrow$ $4u-4v+11=0$

(2) $\Rightarrow$ $4u+6v+2w+d+14=0$ (4) x 10 $\Rightarrow 4u+4v+36=0$

(4) $\Rightarrow$ $8u+6v-2w+d+26=0$----------------

(-) (-) (+) (-) (-) $8u+47=0$

-------------------------- $u= -\frac{47}{8}$

$-4u+4w-12=0$ (7) $\Rightarrow$ $u-w+3=0$

$u-w+3=0$------------(7) $(\frac{-47}{8})-w+3=0$

(2) $\Rightarrow$ $4u+6v+2w+d+14=0$

$w=(-\frac{47}{8})+3=\frac{-23}{8}$

(5) $\Rightarrow$ $4u+10v+6w+d+38=0$ $w=\frac{-23}{8}$

(-) (-) (-) (-) (-)

-------------------------------- (8) $\Rightarrow$ $v+w+6=0$

$-4v-4w-24=0$

$v+(\frac{-23}{8})+6 = 0$

$$v+w+6=0 \text{--------------}(8)$$

$v=\frac{23}{8}-6=-\frac{25}{8}$

$$(6) \Rightarrow \quad 3u-4v+w+8=0$$

$v=-\frac{25}{8}$

$$(7) \Rightarrow \quad u \quad -w+3=0$$

$$4u-4v+11=0 \text{--------------}(9)$$

$$(2) \Rightarrow \quad 4u+6v+2w+d+14=0$$

$\Rightarrow 4(\frac{-47}{8})+6(\frac{-25}{8})+2(\frac{-23}{8})+d+14=0$

$\Rightarrow \frac{-94}{4}-\frac{75}{4}-\frac{23}{4}+d+\frac{56}{4}=0$

$\Rightarrow \quad d=\frac{94+75+23-56}{4}=\frac{136}{4}=34$

$$d=34$$

Finally, we get, $u=\frac{-47}{8}$; $v=\frac{-25}{8}$; $w=\frac{-23}{8}$; $d=34$

$$(1) \Rightarrow \quad x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$\Rightarrow x^2+y^2+z^2+2(\frac{-47}{8})x+2(\frac{-25}{8})y+2(\frac{-23}{8})z+34=0$

$\Rightarrow \quad x^2+y^2+z^2-\frac{47}{8}x-\frac{25}{4}y-\frac{23}{4}z+34=0$

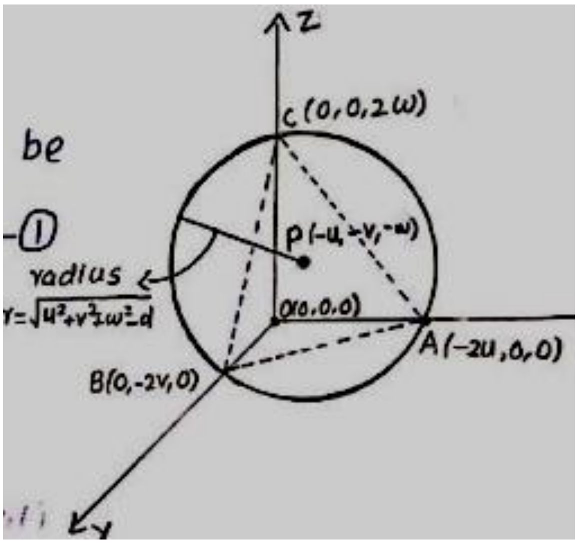
$\Rightarrow 4(x^2+y^2+z^2)-47x-25y-23z+136=0.$

Example 5:

A sphere of constant radius 'k' passes through the origin and meets the axes in A,B,C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$

Solution:

Let the required equation of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ____(1)



Centre of the sphere, P=(-u,-v,-w)

Radius of the sphere, $r = \sqrt{u^2+v^2+w^2-d}$

Given: The sphere passes through the origin O (0,0,0)

$(1) \Rightarrow x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$0+0+0+0+0+0+d=0$

$d=0$

Also, given the radius of the sphere =k (constant)

$\sqrt{(u^2+v^2+w^2-d)} = k$

$\sqrt{u^2+v^2+w^2}$ =k [d=0]

$u^2+v^2+w^2$ = k^2-------(2)

A lie on the x-axis y=0 & z=0

(1) ⇒ $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$x^2+0+0+2ux+0+0+0=0$

$x^2+2ux=0$

$x(x+2u) = 0$

x=0 or x+2u=0

x=0 or x=-2u

x=-2u (x≠0)

The point A=(-2u,0,0)Similarly, B and C lies on the y-axis and z-axis respectively.

B=(0,-2v,0) & C=(0,0,-2w)

Let $G(x_1,y_1,z_1)$ be the centroid of a triangle ΔABC

$$G(x_1,y_1,z_1) = \left(\frac{-2u+0+0}{3}, \frac{0+(-2v)+0}{3}, \frac{0+0+(-2w)}{3}\right)$$

$$G(x_1,y_1,z_1) = \left(\frac{-2u}{3}, \frac{-2v}{3}, \frac{-2w}{3}\right)$$

$$x = -\frac{2u}{3}; \quad y = \frac{-2v}{3}; \quad z = \frac{-2w}{3}$$

$$u = \frac{-3x}{2} \quad v = \frac{-3y}{2}; \quad w = \frac{-3z}{2}$$

substitute the value of u,v,w in equation (2),we get,

$$(2) \Rightarrow \quad u^2+v^2+w^2 = k^2$$

$$\left(\frac{-3x}{2}\right)^2 + \left(\frac{-3y}{2}\right)^2 + \left(\frac{-3z}{2}\right)^2 = k^2$$

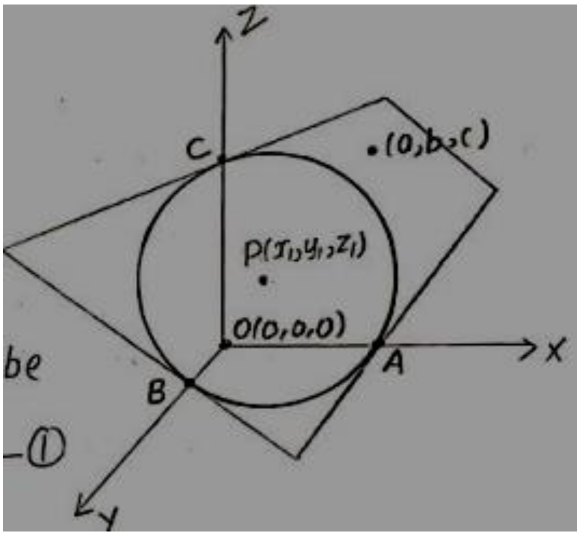
$9x/4 + 9y/4 + 9z/4 = k^2$

$$9(x+y+z) = 4k.$$

Example 6:

A plane passes through a fixed point (a,b,c) and cuts the axes in A,B,C. Show that the locus of the centre of the sphere OABC is a/x +b/y +c/z =2

Solution:



Let P(x,y,z) be the centre of the sphere.

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \dots\dots(1)$$

where, centre of the sphere=(-u,-v,-w) & radius of the sphere $= \sqrt{u^2 + v^2 + w^2 - d}$

A lie on x-axis, y=0 & z=0

(2) ⇒ x+0+0-2xx-0-0=0

$x-2xx_1=0$

$x(x-2x_1) = 0$

$x=0$ or $x=2x_1$

$x=2x_1$ $(x\neq 0)$

The point $A=(2x_1,0,0)$.

Similarly,

the points B and C lies on the y-axis and z-axis respectively.

$B=(0,2y_1,0)$ & $C=(0,0,2z_1)$

(ie) x-intercept=$2x_1$; y-intercept=$2y_1$;& z-intercept=$2z_1$

The equation of the plane passing through A,B,C is

$$\frac{x}{2x_1}+\frac{y}{2y_1}+\frac{z}{2z_1}=1 \text{ (Intercept form) ---------------(3)}$$

Given the plane (1) passing through a fixed point (a,b,c)

$$(3) \Rightarrow \quad \frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=2$$

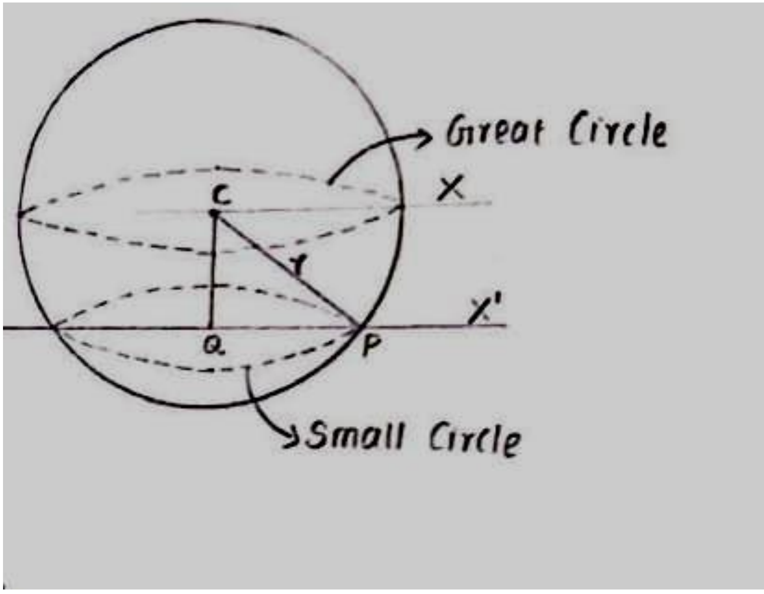
a/x+b/y+c/z = 2

∴ The locus of the centre pf the sphere P(x,y,z) is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2.$$

5.3 Section of a sphere by a plane:

The plane section of a sphere is a circle.



The curve of intersection of a plane and a sphere is a circle.

Let C be the centre of the sphere whose radius is r.

Let P be any point common to the sphere and the plane.

Now, CP = radius of the sphere

Draw CQ perpendicular to the plane X′.

Clearly angle CQP=90°.

From the right-angle triangle CQP, we get,

$$CQ^2 + QP^2 = CP2$$

$$\Rightarrow QP^2 = CP^2 - CQ^2$$

$$\Rightarrow QP^2 = r^2 - CQ^2$$

Since C and Q are fixed points, CQ is a constant.

From, (1) r and CQ are constant

$\Rightarrow$ QP is constant.

Hence the locus of P is a circle whose centre is Q, where Q is the foot of the perpendicular from the centre C of the sphere to the plane X′.

Such a circle is called a small circle on the sphere.

Thus, the section of a sphere by any plane is a circle.

Note:

(1) Q is the foot of the perpendicular from the centre of the sphere to the plane.

(2) The section of the sphere by a plane x passing the centre of the sphere is called a **"Great Circle"**.

(3) The section of the sphere by a plane x' passing through the point Q', the foot of the perpendicular from the centre of the sphere 'C' to the plane X' is called a **small circle**.

5.4 Equation of a circle on a sphere:

We know that the section of the sphere is circle

i.e., The curve of intersection of a sphere by a plane is a circle. So, a circle can be represented by two equations, one being the equation of a sphere and the other is that of a plane.

Thus, the equations $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ and $lx+my+nz=P$ taken together represent a circle.

5.4.1 Equation of a sphere passing through a given circle:

Let the given circle be represented together by the equations of a Spheres and a plane P, where $S = x^2+y^2+z^2+2ux+2vy+2wz+d=0$…… (1)

and $P = lx+my+nz-p=0$ ……(2)

Consider the equation

$$S+kP = (x^2+y^2+2+2ux + 2vy+2wz+d) + k(lx+my+nz-P) = 0, \text{ …….. } (3)$$

where k is a constant.

For different values of k equation (3) represents a sphere.

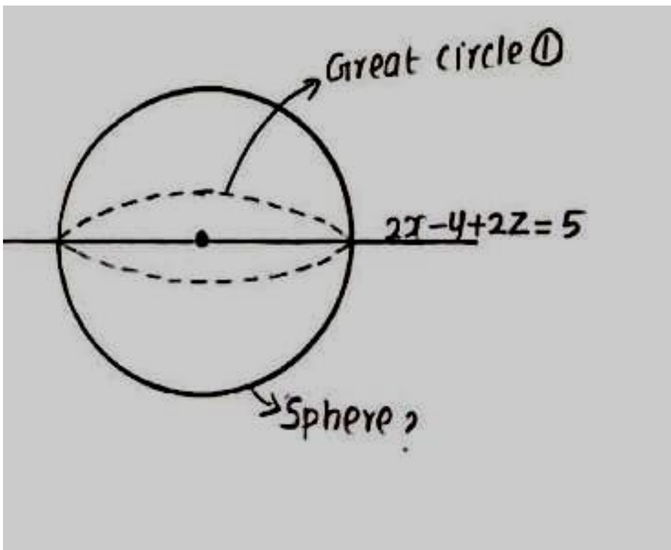
Further,

equation (3) is satisfied by the points common to (1) and (2).

Hence,

equation (3) represents a general sphere passing through the circle given by (1) and (2).

5.4.2 Intersection of Two Spheres:



5.4.3 Intersection of two spheres is a circle.

Let the equations of two spheres be

$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+ d=0$……(1)

$S_2= x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+ d=0$……(2)

Consider the equation

$S_1- S_2 =2x(u- u_1) +2\ y(v - v_1) +2\ z(w - w_1) + (d_1 – d_2))= 0$ …… (3)

Equation (3) is on equation of the first degree.

Equation (3) represents a plane.

Also, it is satisfied by all points common to the spheres (1) and (2).

Hence,

the curve of intersection of the spheres (1) and (2) is same as the curve of intersection of any one of them with the plane (3).

i.e., the curve of intersection of the two spheres is a circle.

Example 1:

Find the equation of the sphere having the circle $x^2 + y^2 + z^2 - 2x + 4y - 6z + 7 = 0, 2x - y + 2z - 5 = 0$ for a great circle.

Solution:

Given $x^2+y^2+z^2-2x+4y-6z+7=0$ and $2x-y+2z-5 = 0$……(1)

is a circle

∴ The equation of any sphere containing the given circle is of the form

$(x^2+y^2+z^2-2x+4y-6z+7) + k(2x-y+2z-5) = 0$

$x^2+y^2+z^2+2(-1+k)\ x +(4-k)\ y+2(-3+k)\ z+(7-5k) =0$

$x^2+y^2+z^2+2\ (-1+k)\ x+\frac{2}{2}\ (4-k)\ y+2(-3+k)\ z+(7-5k)=0$

$\Rightarrow u=-1+k;\ V=\frac{4-k}{2};\ w=(-3+k),\ d=7-5k$

Centre of a sphere (2) = $(-u, -v, -w) = (-(-1+k), -(\frac{4-k}{2}), -(-3+k))$

Centre of a sphere (2) = $(1-k, \frac{k-4}{2}, 3-k)$

Since the given circle (1) is a great circle, the centre of a great circle coincides with the centre of the sphere.

Since the centre of a great circle (1) lies in the plane $2x-y+2z-5=0$, the centre of a sphere (2) lies in the plane $2x-y+z-5=0$.

i.e., $(1-k, \frac{k-4}{2}, 3-k)$ lies in the plane $2x-4+2z-5=0$

$\Rightarrow 2(1-k)-4(\frac{k-4}{2})+2(3-k)-5=0$

$\Rightarrow 2-2k-\frac{k}{2}+2+6+-2k-5 = 0$

$\Rightarrow 5-\frac{9k}{2} = 0$

$\Rightarrow k=\frac{10}{9}$

$\therefore$ (2) $\Rightarrow x^2+y^2+z^2+2\ (-1+\frac{10}{9})x+\frac{2}{2}\ (4-\frac{10}{9})y+2(-3+\frac{10}{9})z+(7-5(\frac{10}{9}))=0$

$\Rightarrow x^2+y^2+z^2+\frac{2}{9}x+\frac{26}{9}y-\frac{34}{9}z+\frac{13}{9}=0$

$\Rightarrow 9(x^2+y^2+z^2) + 2x+26y-34z+13 = 0$

Example 2:

Find the equation of the sphere which passes through the circle $x^2+y^2+z^2-2x-4y=0$, $x+2y+3z=8$ and touches the plane $4x+3y=25$.

Solution:

Given The circle is $x^2+y^2+z^2-2x-4y=0$ and $x+2y+3z = 8$ .....(1)

and the plane is $4x+3y=25$......(2)

The equation of any sphere containing the given circle (1) is of the form

$(x^2+y^2+z^2-2x-4y) + k(x+2y+3z-8) = 0$

$x^2+y^2+z^2+(-2+k)\,x+2(-2+k)\,y+3kz-8k = 0$ .....(3)

$u = \frac{-2+k}{2}$; $v = -2 + k, w = \frac{3k}{2}$; $d= -8k$

Centre of a sphere (1), C = (-u, -v,-w)

$$= (-(\frac{-2+k}{2}),-(-2+k),- (\frac{3k}{2}))$$

i.e., $C = (\frac{2-k}{2}), 2-k, (\frac{3k}{2}))$

Radius of a sphere equation (1), $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{(\frac{2-k}{2})^2 + 2 - k^2 + (\frac{3k}{2})^2 - (-8k)}$$

$$= \left\{\frac{4+k^2-4k}{4} + 4 + k^2 - 4k + \frac{9k^2}{4} + 8k\right\}^{\frac{1}{2}}$$

$$= \frac{\{4+k^2-4k+16+4k^2-16k+9k^2+32k\}^{\frac{1}{2}}}{2}$$

$$= \left\{\frac{14k^2+12k+20}{4}\right\}^{\frac{1}{2}}$$

$$=\left\{\frac{7k^2+6k+10}{2}\right\}^{\frac{1}{2}}$$

Given the circle (1) touch the plane (2) 4x+3y=25.

⇒ The perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere.

i.e., Perpendicular distance from $\left(\frac{2-k}{2}, 2\text{-k}, \frac{3k}{2}\right)$ to the plane 4x+3y-25=0

= Radius of the sphere.

$$\Rightarrow \left(\frac{4\left(\frac{2-k}{2}\right)+3(2-\text{k})+0\left(-\frac{3k}{2}\right)-25}{\sqrt{4^2+3^2+0^2}}\right) = \left\{\frac{7k^2+6k+10}{2}\right\}^{\frac{1}{2}}$$

$$\Rightarrow \frac{2(2-\text{k})+3(2-\text{k})-25}{\sqrt{25}} = \left\{\frac{7k^2+6k+10}{2}\right\}^{\frac{1}{2}}$$

$$\Rightarrow \frac{4-2k+6-3k-25}{5} = \left\{\frac{7k^2+6k+10}{2}\right\}^{\frac{1}{2}}$$

$$\Rightarrow \frac{-5k-15}{5} = \left\{\frac{7k^2+6k+10}{2}\right\}^{\frac{1}{2}}$$

$$\Rightarrow -(k+3) = \left\{\frac{7k^2+6k+10}{2}\right\}^{\frac{1}{2}}$$

$$\Rightarrow (k+3)^2 = \left\{\frac{7k^2+6k+10}{2}\right\}$$

$\Rightarrow 2(k^2+9+6k) = 7k^2+6k+10$

$\Rightarrow 2k^2+18+12k = 7k^2+6k+10$

$\Rightarrow 5k^2-6k-8=0$

$\Rightarrow (5k+4)(k-2)=0$

$\Rightarrow$ k= $\frac{-4}{5}$ or k=2.

When k = $\frac{-4}{5}$,

(3) $\Rightarrow x^2+y^2+z^2 + (-2+ (\frac{-4}{5}))\,x+(-2 +(\frac{-4}{5}))\,y+3(\frac{-4}{5})z-8(\frac{-4}{5})=0$

$\Rightarrow x^2+y^2+z^2 -\frac{14}{5}x-\frac{28}{5}y-\frac{14}{5}z+(\frac{32}{5})=0$

$\Rightarrow x^2+y^2+z^2+\frac{1}{5}(-14x-28y-12z+32) = 0$

$\Rightarrow 5(x^2+y^2+z^2)-14x-28y-12z+32=0.$

When k=2,

(3) $\Rightarrow x^2+y^2+z^2 + (-2+ 2)\,x+(-2 +2)\,y+3(2)z-8(2)=0$

$\Rightarrow x^2+y^2+z^2+6z-16 = 0$

Hence, the required spheres are

$x^2+y^2+z^2+6z-16 = 0$ and $5(x^2+y^2+z^2)-14x-28y-12z+32=0.$

Example 3:

The plane ABC, whose equation $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, meets the axes in A, B, C. Find the equation to the circum circle of the triangle ABC and obtain the co-ordinates of its centre and radius

Solution:

Given the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$…… (1)

Given the plane (1) meets the axes in A, B, C respectively

$\therefore$ A = (a, 0, 0); B = (0, b, 0); C = (0, 0, c).

Let O = (0, 0, 0) be the origin.

Let the equation of the sphere OABC be $x^2+y^2+z^2+2ux+2vy+2wz+ d=0$.

Since the points O, A, B and C lies in this sphere, we get,

O $\Rightarrow$ 0 + 0 + 0 + 0 + 0 + 0 + d = 0 then d = 0

A $\Rightarrow$ a^2+ 0 + 0 + 2au + 0 + 0 + 0 = 0 then 2u $=\frac{-a^2}{a}$

2u = - a

B $\Rightarrow$ 0+ b^2+ 0 + 0 + 2bv + 0 + 0 = 0 then 2v =-b

C $\Rightarrow$ 0 + 0 + c^2 + 0 + 0 + 2cw + 0 = 0 then 2w = -c

The equation of the sphere OABC becomes $x^2+y^2+z^2+2ux+2vy+2wz+ d=0$

$x^2+y^2+z^2-ax-by-cz+ d=0$ ……… (2)

Let P and r be the centre and radius of the sphere.

$\therefore$ P = (- u, - v, - w) = $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ and

$$r = \sqrt{u^2+v^2+w^2-d}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}-0}$$

$$= \sqrt{\frac{a^2+b^2+c^2}{4}}$$

Also, the equation of the circumcircle of a triangle ABC is

$x^2+y^2+z^2-ax-by-cz+ d=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Let Q and R be the centre and radius of the circumcircle (3).

To find: Q and R

(1) To find the centre of the circumcircle of equation (3) is (Q)

Clearly,

the centre of the circumcircle is the foot of the perpendicular from

$$P = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \text{ to the plane (1).}$$

Now,

the equation of the perpendicular line to the plane (1) through the point

$P = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$ is $\frac{x-\frac{a}{2}}{\frac{1}{a}} = \frac{y-\frac{b}{2}}{\frac{1}{b}} = \frac{z-\frac{c}{2}}{\frac{1}{c}}$.

$$\text{Let } \frac{x-\frac{a}{2}}{\frac{1}{a}} = \frac{y-\frac{b}{2}}{\frac{1}{b}} = \frac{z-\frac{c}{2}}{\frac{1}{c}} = \lambda \text{ (say)}$$

⇒ The co-ordinates of any point on this line are of the form $\left(\frac{a}{2} + \frac{\lambda}{a}, \frac{b}{2} + \frac{\lambda}{b}, \frac{c}{2} + \frac{\lambda}{c}\right)$.

Let Q = $\left(\frac{a}{2} + \frac{\lambda}{a}, \frac{b}{2} + \frac{\lambda}{b}, \frac{c}{2} + \frac{\lambda}{c}\right)$ (Since Q lies on this line)

Since this point lies on the plane, we get from (1), we get,

$$(1) \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\Rightarrow \left(\frac{1}{a}\left(\frac{a}{2} + \frac{\lambda}{a}\right) + \frac{1}{b}\left(\frac{b}{2} + \frac{\lambda}{b}\right) + \frac{1}{c}\left(\frac{c}{2} + \frac{\lambda}{c}\right)\right) = 1$$

$$\Rightarrow \frac{1}{2} + \frac{\lambda}{a^2} + \frac{1}{2} + \frac{\lambda}{b^2} + \frac{1}{2} + \frac{\lambda}{c^2} = 1$$

$$\Rightarrow \lambda\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{-1}{2}$$

$$\Rightarrow \lambda = \frac{-1}{2(a^{-2}+b^{-2}+c^{-2})}$$

$\therefore$ Q $= \left(\frac{a}{2}+\frac{\lambda}{a}, \frac{b}{2}+\frac{\lambda}{b}, \frac{c}{2}+\frac{\lambda}{c}\right)$ becomes,

$$Q = \left(\frac{a}{2}+\frac{1}{2a\,(a^{-2}+b^{-2}+c^{-2})}, \frac{b}{2}-\frac{1}{2b\,(a^{-2}+b^{-2}+c^{-2})}, \frac{c}{2}-\frac{1}{2c\,(a^{-2}+b^{-2}+c^{-2})}\right)$$

$$= \left(\frac{a\,(a^{-2}+b^{-2}+c^{-2})-1/a}{2\,(a^{-2}+b^{-2}+c^{-2})}, \frac{b\,(a^{-2}+b^{-2}+c^{-2})-1/b}{2\,(a^{-2}+b^{-2}+c^{-2})}, \frac{c\,(a^{-2}+b^{-2}+c^{-2})-1/c}{2\,(a^{-2}+b^{-2}+c^{-2})}\right)$$

$$= \left(\frac{a\,(a^{-2}+b^{-2}+c^{-2})-a^{-2}}{2\,(a^{-2}+b^{-2}+c^{-2})}, \frac{b\,(a^{-2}+b^{-2}+c^{-2})-b^{-2}}{2\,(a^{-2}+b^{-2}+c^{-2})}, \frac{c\,(a^{-2}+b^{-2}+c^{-2})-c^{-2}}{2\,(a^{-2}+b^{-2}+c^{-2})}\right)$$

$$\therefore \quad Q = \left(\frac{a\,(b^{-2}+c^{-2})}{2\,(a^{-2}+b^{-2}+c^{-2})}, \frac{b\,(a^{-2}+c^{-2})}{2\,(a^{-2}+b^{-2}+c^{-2})}, \frac{c\,(a+b^{-2})}{2\,(a^{-2}+b^{-2}+c^{-2})}\right)$$

Hence the centre of the circumcircle of the triangle ΔABC is

$$Q = \left(\frac{a/2\,(b^{-2}+c^{-2})}{a^{-2}+b^{-2}+c^{-2}}, \frac{b/2\,(b^{-2}+c^{-2})}{a^{-2}+b^{-2}+c^{-2}}, \frac{c/2\,(b^{-2}+c^{-2})}{a^{-2}+b^{-2}+c^{-2}}\right).$$

(ii) To find the radius of the circumcircle (3) is (R)

Let d be the perpendicular distance from the point P = (a/2, b/2, c/2) to the plane (1) $x/a + y/b + z/c = 1$

$$\therefore \quad d = \frac{\frac{1}{a}\left(\frac{a}{2}\right)+\frac{1}{b}\left(\frac{b}{2}\right)+\frac{1}{c}\left(\frac{c}{2}\right)-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow d = \frac{\frac{3}{2}-1}{\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right\}^{1/2}}$$

$$\Rightarrow d = \frac{\frac{1}{2}}{\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right\}^{1/2}}$$

Also, we have the centre of the sphere (2) is

$$r = \sqrt{\frac{a^2+b^2+c^2}{4}}$$

W.K.T,

$$R^2 = r^2 - d^2$$

$$R^2 = \frac{a^2+b^2+c^2}{4} - \frac{1}{4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

$$= \frac{a^2+b^2+c^2}{4} - \frac{1}{4\left(\frac{b^2c^2+a^2c^2+b^2a^2}{a^2b^2c^2}\right)}$$

$$= \frac{a^2+b^2+c^2}{4} - \frac{a^2b^2c^2}{4\,(b^2c^2+a^2c^2+b^2a^2)}$$

$$= \frac{(a^2+b^2+c^2)\left(b^2c^2+a^2c^2+b^2a^2\right)-a^2b^2c^2}{4\ (b^2c^2+a^2c^2+b^2a^2)}$$

$$= \frac{(a^2+b^2)\left(b^2c^2+a^2b^2\right)+c^2\left(b^2c^2+a^2c^2+b^2a^2\right)-a^2b^2c^2}{4\ (b^2c^2+a^2c^2+b^2a^2)}$$

$$= \frac{(a^2+b^2)\left(b^2c^2+a^2b^2\right)+\left(b^2c^2+a^2c^2+b^2a^2\right)-a^2b^2c^2}{4\ (b^2c^2+a^2c^2+b^2a^2)}$$

$$= \frac{(a^2+b^2)\left(b^2c^2+a^2b^2+a^2c^2\right)+c^4(a^2+b^2)}{4\ (b^2c^2+a^2c^2+b^2a^2)}$$

$$= \frac{(a^2+b^2)\left(b^2c^2+a^2b^2+a^2c^2\right)+c^4}{4\ (b^2c^2+a^2c^2+b^2a^2)}$$

$$= \frac{(a^2+b^2)\left(b^2(c^2+a^2)+c^2(a^2c^2)\right)}{4\ (b^2c^2+a^2c^2+b^2a^2)}$$

$$\therefore R^2 = \frac{(a^2+b^2)\,(b^2+c^2)\,(c^2+a^2))}{4\ (b^2c^2+\,a^2c^2+\,b^2a^2)}$$

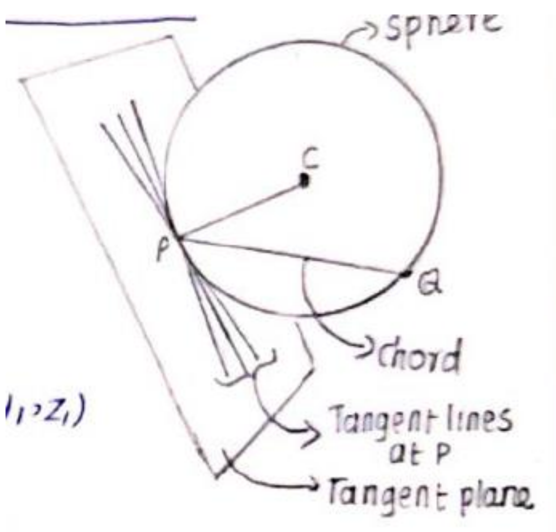
$$\Rightarrow R = \frac{1}{2}\left\{\frac{(a^2+b^2)(b^2+c^2)(c^2+a^2)}{b^2c^2+a^2c^2+a^2b^2}\right\}^{\frac{1}{2}}$$

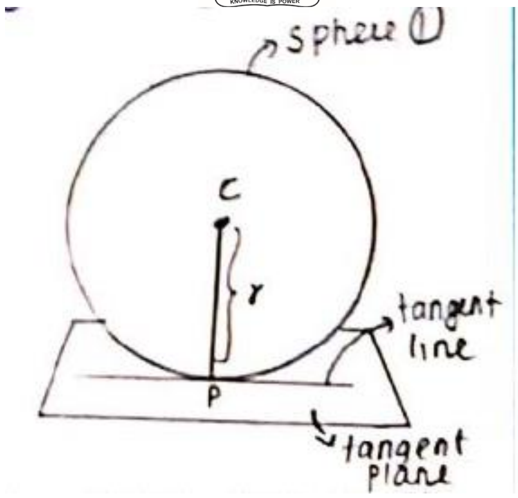
5.5 Tangent Plane:

Definition:

The straight line joining two points P and Q on a surface is called a chord of the surface When Q moves along the surface and ultimately coincides with P the limiting position of PQ touches the surface at P and is called a tangent line of the surface

In the case of sphere with centre C there are many tangent lines at a point P on it, all of them being perpendicular to the radius (P All these tangents lie on the plane through P perpendicular to CP. This plane is called the tangent plane of the sphere at P.





5.5.1 The equation of the Tangent Plane to a Sphere:

The equation of the tangent plane to a sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at the point P (x_1, y_1, z_1) is $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

Proof:

Let the given sphere be Sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ …… (1)

& the point P (x_1, y_1, z_1) be any point on this sphere (1).

$\therefore$ the centre of the given sphere (1) is $C = (-u, -v, -w)$

Since P lies on the sphere,

$$(1) \Rightarrow x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \text{…….. (2)}$$

The tangent plane at P is the plane passes through the point P (x_1, y_1, z_1) and perpendicular to the radius CP.

$\because$ $C = (-u, -v, -w)$ and P (x_1, y_1, z_1), Directional ratios of CP = x_1+u, y_1+v, z_1+w,

$\therefore$ The equation of the tangent plane through the point P (x_1, y_1, z_1) and having CP as its normal is

$$(x_1+u)(x-x_1) + (y_1+v)(y-y_1) + (z_1+w)(z-z_1) = 0.$$

$$\Rightarrow xx_1 - x_1^2 + ux - u x_1 + yy_1 - y_1^2 + vy - v y_1 + zz_1 - z_1^2 + wz - wz_1 = 0.$$

$$\Rightarrow xx_1 + yy_1 + zz_1 + ux + vy + wz - x_1^2 - y_1^2 - z_1^2 - u x_1 - v y_1 - wz_1 = 0 \ldots..(3)$$

Adding (2) and (3), we get,

$$xx_1 + yy_1 + zz_1 + ux + vy + wz - x_1^2 - y_1^2 - z_1^2 + u x_1 + v y_1 + wz_1 + d = 0.$$

$$\Rightarrow xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$$

Hence the equation of the tangent plane to the sphere $x^2+y^2+z^2+aux+2vy+2wz+ d = 0$

at the point P (x_1, y_1, z_1) is

$$xx_1 + yy_1 + zz_1 + u (x + x_1) + v (y + y_1) + w (z + z_1) + d = 0.$$

Note 1:

Rule for writing down the equation of the tangent plane at the point (x_1, y_1, z_1) to given sphere whose equation is given in standard form is

change x^2 into xx_1, y^2 into yy_1, z^2 into zz_1,
$2x$ into $(x + x_1)$, $2y$ into $(y + y_1)$, $2z$ into $(z + z_1)$,
keep the constant term d without any change.

Note 2:

The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x_1, y_1, z_1) on it is $xx_1 + yy_1 + zz_1 = a^2$.

5.6 Angle of intersection of two spheres:

The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point.

Since the angle between two tangent planes at the common point is same as the angle between their normal at that point, the angle between the two spheres is same as the angle between the radii of the two spheres at the common point.

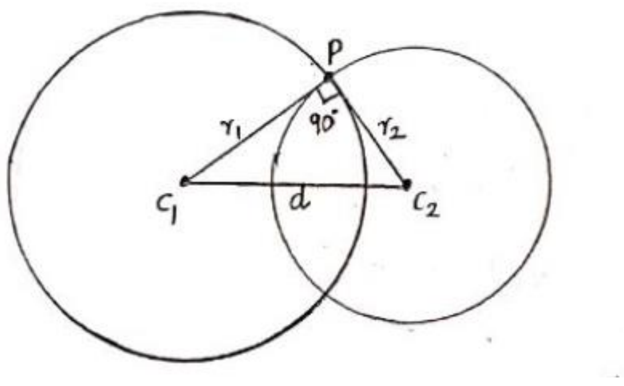
Also, we note that the angle of intersection at every common point of the sphere is the same.

5.6.1 Orthogonal Spheres:

Two spheres are said to cut each other orthogonally if the tangent planes at a point of intersection are at right angles.

5.7 Condition for the orthogonality:

5.7.1 Condition for the Two Spheres to cut orthogonally:



If two spheres cut orthogonally at P, their radius through P, being perpendicular to the tangent planes at P, will also be at right angles.

Consider the two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\& \; x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad \ldots\ldots\ldots\ldots\ldots..(2)$$

Let these spheres (1) & (2) cut orthogonally at 'P'.

For the sphere (1),

Centre $C_1 = (-u_1,-v_1,-w_1)$

Radius $r_1= \sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}$ …………..(3)

For the sphere (2),

Centre $C_2 = (-u_2,-v_2,-w_2)$

Radius $r_1= \sqrt{u_2^2 + v_2^2 + w_2^2 - d_2}$ …………..(4)

Since they cut orthogonally at P, $\angle C_1PC_2= 90^o$.

∴ From the right-angle triangle $\triangle C_1PC_2$, we get,

$$C_1C_2= d =\sqrt{(u_1 - u_2)^2 \; + \; (v_1 - v_2)^2 \; + \; (w_1 - w_2)^2}$$

$$(C_1C_2)^2= C_1P^2+ C_2P^2$$

$$\Rightarrow d^2= r_1^2+ r_2^2 \quad \ldots\ldots\ldots\ldots. (6)$$

Substitute (3), (4), (5) in (6), we get,

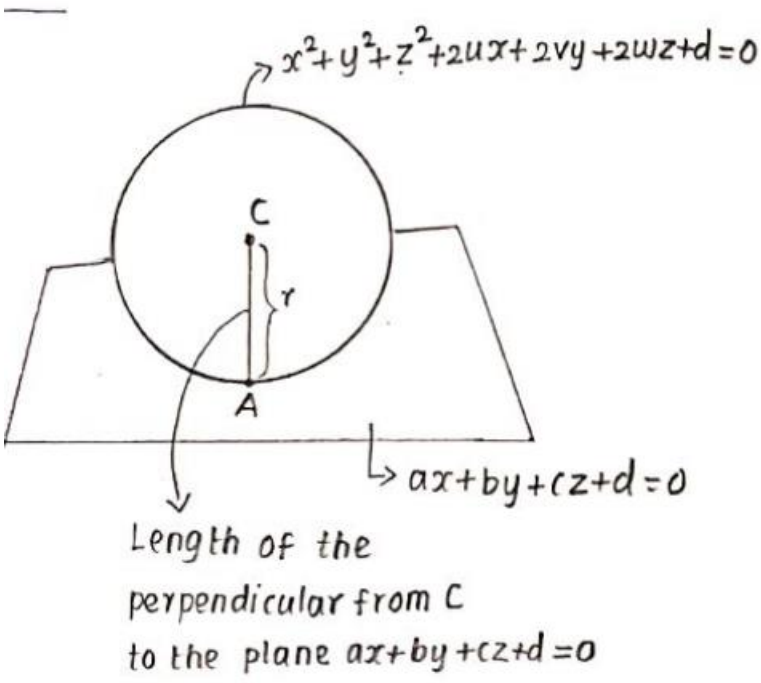
$$(u_1 - u_2)^2 \; + (v_1 - v_2)^2 \; + \; (w_1 - w_2)^2 = \; (u_1^2 + v_1^2 + w_1^2 - d_1)+(u_2^2 + v_2^2 + w_2^2 - d_2)$$

$$\Rightarrow 2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$$

Hence the condition for two spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ and $x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ to cut orthogonally is $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$.

5.7.2 Find the condition that the plane $ax+by+cz+d=0$ may touches the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

A plane will touch a sphere if the length of the perpendicular from the centre of the sphere to the plane is equal to the radius of the sphere.



Given the plane equation is $ax+by+cz+d=0$……. (1)

& the sphere is $x^2+y^2+z^2+aux+2vy+2wz+d=0$ ……. (2)

Centre of the sphere (2) is (-u, -v, -w)

Radius of the sphere (2) is $\sqrt{u^2+v^2+w^2-d} = \sqrt{4+1+1+3} = \sqrt{9} = 3$

Also,

The length of the perpendicular from the point $(-u, -v, -w)$ to the plane

$ax + by + cz + d = 0$

$$= \pm\left(\frac{a(-u)+b(-v)+c(-w)+d}{\sqrt{a^2+b^2+c^2}}\right)$$

$$= \frac{-au-bv-cw+d}{\sqrt{a^2+b^2+c^2}} \text{ (Take '+' sign)}$$

Given the length of the perpendicular = Radius of the sphere

$$\Rightarrow \frac{-au-bv-cw+d}{\sqrt{a^2+b^2+c^2}} = \sqrt{u^2+v^2+w^2-d}$$

$$\Rightarrow -(au-bv-cw+d) = \sqrt{a^2+b^2+c^2}\sqrt{u^2+v^2+w^2-d}$$

$$\Rightarrow (au-bv-cw+d)^2 = (a^2+b^2+c^2)(u^2+v^2+w^2-d)$$

Hence the condition that the plane $ax + by + cz + d = 0$ touches the sphere

$x^2+y^2+z^2 + 2ux + 2vy + 2wz+d = 0$ is

$$(au-bv-cw+d)^2 = (a^2+b^2+c^2)(u^2+v^2+w^2-d).$$

Example 1:

Show that the plane $2x - y - 2z = 16$ touches the sphere $x^2+y^2+2^2-4x+2y+2z-3=0$ and find the point of contact.

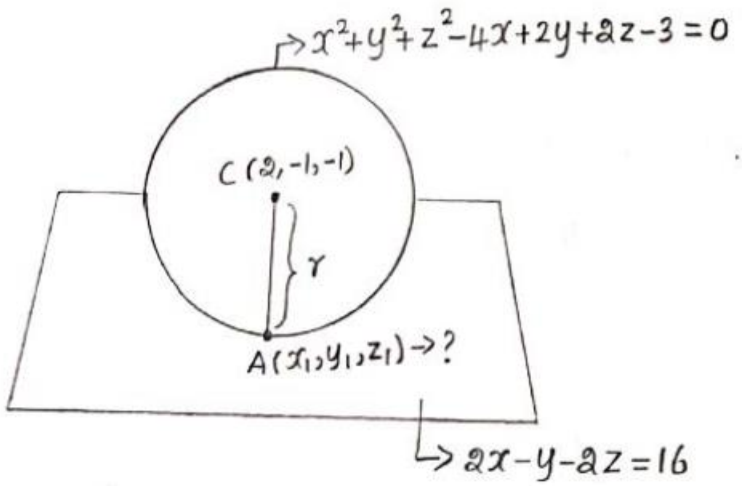
Solution:

Given the plane is $2x - y - 2z = 16$………(1)

and the sphere is $x^2+y^2+2^2-4ux+2y+2z-3=0$………(2)

⇒ The centre of the sphere, C = (-u, -v, -w) = $(2, -1, -1)$

and radius of the sphere is $\sqrt{u^2 + v^2 + w^2 - d}$ = $\sqrt{4+1+1+3}$ = $\sqrt{9}$ = 3



Now,

The length of the perpendicular from $C(2, -1, -1)$ to the $2x - y - 2z - 16 = 0$

$$=\pm\left(\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right)$$

$$=\pm\left(\frac{a(-u)+b(-v)+c(-w)+d}{\sqrt{a^2+b^2+c^2}}\right)$$

$$=\pm\left(\frac{2(2)-(-1)-2(-1)-16}{\sqrt{2^2+(-1)^2+(-2)^2}}\right) = \pm\left(\frac{-9}{3}\right) = \pm(-3) = 3.$$

∴ Length of the perpendicular = Radius of the sphere.

Hence the plane touches the sphere.

Let A (x_1, y_1, z_1) be the point of contact.

Here the point of contact 'A' is the foot of the perpendicular from the centre C(2,-1,-1) of the sphere equation (2) on the plane (1).

Now,

The directional ratios of the normal to the plane (1) = ((2,-1,-2)

⇒ the directional ratios of the line CA = (2,-1,-2) [Since the line CA parallel to the normal of the plane equation (1)]

∴ Equation of the line CA passing through the point (2,-1,-1) with the direction ratio 2,-2,1 is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\Rightarrow \frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+1}{-2} \text{ .... (3)}$$

$$\text{Let } \frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+1}{-2} = k \text{ (say)}$$

⇒ x= 2 + 2k; y = - 1 - k, z=-1-2k

Any point on the line (3) is of the form (2 + 2k, - 1 – 1k, - 1 – 2k)

Since A (x_1, y_1, z_1) lies on the line (3),

A (x_1, y_1, z_1) = (2 + 2k, - 1 – 1k, - 1 – 2k) .... (4)

Since A (x_1, y_1, z_1) lies in the plane (1) 2x-y-2z = 16, also

(1) ⇒ 2(2 + 2k) - (- 1 - k) - 2(- 1 – 2k) = 16

⇒ 4+4k+1+k+2+4k=16

⇒ 9k+7=16

⇒ 9k=9

⇒ k=1

(4) ⇒ A $(x_1, y_1, z_1) = (2 + 2(1), - 1 - 1, - 1 - 2) = (4, -2, -3)$.

∴ The point of the contact = (4, -2, -3)

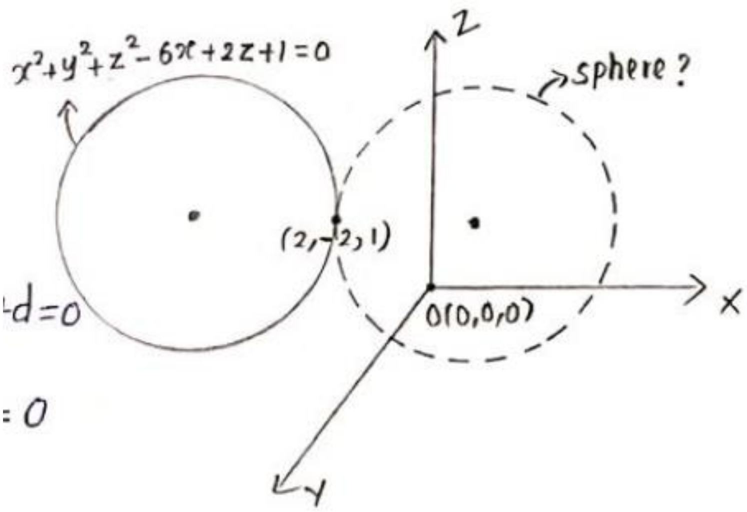
Example 2:

Find the equation of a sphere which touches the sphere $x^2+y^2+z^2-6x+2z+1=0$ at the point (2, -2, 1) and passes through the origin.

Solution:

Given the sphere is $x^2+y^2+z^2-6x+2z+1=0$ …………. (1)

$\Rightarrow u=-3; v=0; w=1; d= 1.$



Now,

The tangent plane to the sphere (1) at (2, -2, 1) is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

$$\Rightarrow 2x - 2y + z -3(x + 2) + o(y - 2) + 1(z + 1) + 1 = 0$$

$$\Rightarrow 2x - 2y + z - 3x - 6 + z + 1 + 1 = 0$$

$\Rightarrow - x - 2y + 2z - 4 = 0$

$\Rightarrow x + 2y - 2z + 4 = 0$

$\therefore$ The required sphere is of the form

$$(x^2+y^2+z^2-6x+2z+1) + k\,(x + 2y - 2z + 4) = 0 \;\ldots\ldots\ldots\; (2)$$

Given the sphere (2) passes through the origin (0,0,0).

$(2) \Rightarrow (0 + 0 + 0 - 0 + 0 + 1) + k\,(0 + 0 - 0 + 4) = 0$

$\Rightarrow 1+4k = 0$

$\Rightarrow k = \frac{-1}{4}$

$\therefore$ The equation of the required sphere (2) becomes

$(2) \Rightarrow (x^2+y^2+z^2-6x+2z+1) + \frac{-1}{4}(x + 2y - 2z + 4) = 0$

$\Rightarrow 4\,(x^2+y^2+z^2-6x+2z+1) - (x + 2y - 2z + 4) = 0$

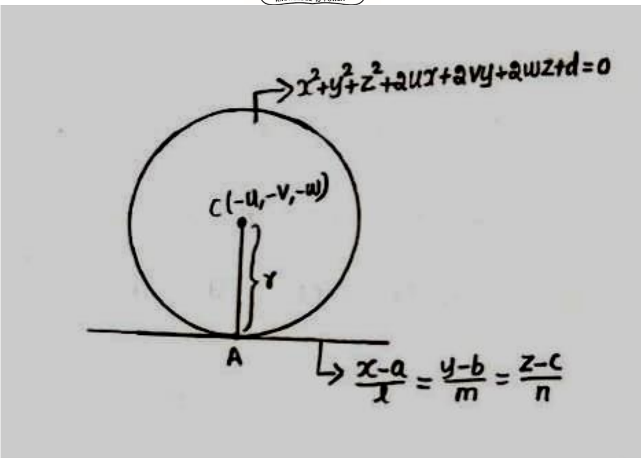
$\Rightarrow 4\,(x^2+y^2+z^2) - 24x+8z+4-x-2y+2z-4 = 0$

$\Rightarrow 4\,(x^2+y^2+z^2) - 25x-2y+10z = 0,$

which is the required equation of the sphere.

Example 3:

Find the condition that the line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$, where $l^2+m^2+n^2 = 1$ should touch the sphere $x^2+y^2+2^2+2ux+2vy+2wz+ d=0$. Show that there are two spheres through the points (0,0,0), (2a,0,0), (0,2b.0) which touch the above the line and that the distance between their centres is $\frac{2}{n^2}(c^2 - (a^2 + b^2+c^2)^{\frac{1}{2}}$



Solution:

Given the line is $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$, ... (1)

where $l^2+m^2+n^2 = 1$.

and the sphere is $x^2+y^2+2^2+2ux+2vy+2wz+ d=0$. ……….. (2)

⇒ Centre of the sphere, C= (-u, -v, -w)

and Radius of the sphere is r = $\sqrt{u^2+v^2+w^2-d}$

Let A be the point of intersection of the line and the sphere.

Let $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ = k (say)

$\Rightarrow x = a + lk;\ y = b + mk\ ;\ z = c + nk.$

Any point on the line equation (1) is of the form $(a + lk, b + mk, c + nk)$.

Since A lies on the line (1), A = $(a + 1k, b + mk, c + nk)$.

Since A lies on the sphere (2) also,

(2) ⇒ $x^2+y^2+2^2+2ux+2vy+2wz+ d=0$.

⇒ $(a + 1k)^2 + (b + mk)^2 + (c + nk)^2 + 2u(a + 1k) + 2v(b + mk)+2w(c + nk) + d=0$

$\Rightarrow a^2+l^2k^2+2alk+b^2+m^2k^2+2bmk+c^2+n^2k^2+2cnk+2ua+2ulk+2vb$
$+2vmk+2wc+2wnk+d=0$

$\Rightarrow k^2(l^2+m^2+n^2)+2k(al+bm+cn+ul+vm+wn)+a^2+b^2+c^2+2ua+2vb+2wc+d=0.$

$\Rightarrow k^2+2k[l(a+u)+m(b+v)+n(c+w)]+a^2+b^2+c^2+2ua+2vb+2wc+d=0.$

Here $A=1$; $B=2[l(a+u)+m(b+v)+n(c+w)]$; $C=a^2+b^2+c^2+2ua+2vb+2wc+d=0.$

Given the line touches the sphere (2),

Discriminant, $D=B^2-4AC=0.$

$$\Rightarrow B^2=4AC$$

$\Rightarrow\{2[l(a+u)+m(b+v)+n(C+w)]\}^2=4(a^2+b^2+c^2+2ua+2vb+2wc+d)$ …... (2)

$\Rightarrow[\{l(a+u)+m(b+v)+n(c+w)]^2=a^2+b^2+c^2+2ua+2vb+2wc+d,$ ………… (3)

which is the condition that the line equation (1) touches the sphere (2).

Let the equation of the sphere passing through the points (0,0,0), (2a,0,0), (0,2b,0) be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$\Rightarrow 0+0+0+0+0+0+d=0.$

$\Rightarrow d=0.$

$\Rightarrow (2a)^2+0+0+2u(2a)+0+0+0=0.$

$\Rightarrow u=-a$

$\Rightarrow 0+(2b)^2+0+0+2v(2b)+0+0=0.$

$\Rightarrow v=-b$

The equation of the sphere becomes $x^2+y^2+z^2-2ax-2by+2wz+d=0$ ……… (4)

If this sphere (4) touches the line (1), then it satisfies condition (3).

$\therefore$ (3) $\Rightarrow [l(a-a) + m(b-b)+n(c+w)]^2 = a^2+b^2+c^2+2(-a)a+2(-b)b+2wc+0.$

$\Rightarrow n^2 (c+w)^2 = a^2+b^2+c^2-2a^2-2b^2+2wc$

$\Rightarrow n^2 (c^2+w^2+2wc) = -a^2-b^2+c^2+2wc$

$\Rightarrow n^2c^2+n^2w^2+2wcn^2 + a^2+b^2- c^2-2wc = 0$

$\Rightarrow n^2w^2+2(n^2-1)cw + a^2+b^2- c^2+ n^2c^2 = 0$ ……….. (5)

Clearly it is a quadratic equation in 'w'.

$\Rightarrow$ There are two roots of 'w' satisfying this equation (5).

$\therefore$ There are two spheres touching the line (1).

Let the roots of the equation (5) be $w_1 \neq w_2$.

i.e., $w = w_1$ or w_2

Now,

Sum of the roots,

$$w_1 + w_2 = \frac{-2(n^2-1)c}{n^2}$$

& Product of the roots,

$$w_1w_2 = \frac{a^2+b^2-c^2+n^2c^2}{n^2}$$

W.K.T,

In general,

the centre of a sphere = (- u, - v, - w).

Here u = 0, v= -b, w = w_1 *or* w_2

∴ The centres of the two spheres touches the line (1) is (a, b, - w_1}) & (a, b, - w_2)

∴ Distance between the centres of the two spheres touches the line (1)

$$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$=\sqrt{(a-a)^2+(b-b)^2+(-w_2-(-w_1))^2}$$

$$=\sqrt{(w_1-w_2)^2}$$

$$=\sqrt{(w_1+w_2)^2-4w_1w_2}$$

$$=\sqrt{\left(\frac{-2(n^2-1)c}{n^2}\right)^2-4\left(\frac{\mathrm{a^2+b^2-c^2+n^2c^2}}{n^2}\right)}$$

$$=\sqrt{\left(\frac{4c^2(n^2-1)^2}{n^4}\right)-\left(\frac{4(\mathrm{a^2+b^2-c^2+n^2c^2})}{n^2}\right)}$$

$$=\frac{2}{n^2}\{c^2+(n^4+1-2n^2)-\mathrm{n^2a^2-n^2b^2+n^2c^2-n^4c^2}\}^{\frac{1}{2}}$$

$$=\frac{2}{n^2}\{c^2-\mathrm{n^2a^2-n^2b^2-n^2c^2}\}^{\frac{1}{2}}$$

$$=\frac{2}{n^2}\left\{c^2-\mathrm{n^2(a^2-b^2+c^{2)}}\right\}^{\frac{1}{2}}$$

Hence the distance between the centers = $\frac{2}{n^2}\left\{c^2-\mathrm{n^2(a^2-b^2+c^{2)}}\right\}^{\frac{1}{2}}$.

Exercises:

1.Find the radius and the co-ordinates of the center of each of the following spheres:

(i) $x^2+y^2+z^2-2x+6y+4z-35=0$.

(ii) $x^2+y^2+z^2-6x-2y-4z-11=0$.

(iii) $16x^2 + 16y^2 + 16z^2 - 16x - 8y - 16z - 55 = 0$.

(iv) $2x^2 + 2y^2 + 2z^2 + 8x - 8y - 6z - 1 = 0$.

2. Find the equation of the following spheres:

(i) Centre at $(1,2,3)$; radius = 4.

(ii) Centre at $(-6,-2,3)$; radius = 5.

(iii) Centre at $\left(\frac{-1}{3}, \frac{2}{3}, \frac{1}{3}\right)$; radius = 1.

3.Find the equation of the sphere whose center is at $(2,3,0)$ and which passes through $(1,0,2)$.

4. Find the equation of the sphere through the four points and determine its radius:

(i) $(0,0,0), (a,0,0), (0,b,0), (0,0,c)$.

(ii) $(0,1,3), (1,2,4), (2,3,1), (3,0,2)$.

5. Obtain the equation of the sphere circumscribing the tetrahedron whose faces are $x = 0, y = 0, z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

6. Show that the spheres $x^2 + y^2 + z^2 = 25$; $x^2 + y^2 + z^2 - 18x - 24y - 40z = -225$ touch and find the co-ordinate of their common point.

7. Find the condition that the plane $lx + my + nz = p$ should touch the points sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

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